Invariant functions on commuting schemes via Langlands duality

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Overview

Joint work with Penghui Li and David Nadler (arXiv:2301.02618)
Plan:

1. Statements (Lie theory and commutative algebra)
2. Proof outline (Betti geometric Langlands, cocenter of affine Hecke category)

Statements

Notation:

- G: connected reductive group over $\mathbb{C}$.
- $\mathcal{C}_G^2$ (commuting scheme for G): pairs $(g_1, g_2) \in G \times G$ satisfying the equation $g_1 g_2 = g_2 g_1$.
- G acts on $\mathcal{C}_G^2$ by simultaneous conjugation.

Goal:

Understand G-invariant regular functions on $\mathcal{C}_G^2$, i.e., $\mathcal{O}(\mathcal{C}_G^2)^G$.

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Toy model

Consider G-conjugation invariant functions on G.

Let $T \subset G$ be a maximal torus, W the Weyl group.

Theorem (Chevalley restriction theorem)

Restriction to T gives an isomorphism of $\mathbb{C}$-algebras

$$\mathcal{O}(G)^G \xrightarrow{\sim} \mathcal{O}(T)^W.$$

Example: $G = \mathrm{GL}_n$,

$$\mathcal{O}(G)^G \cong \mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]^{S_n} \cong \mathbb{C}[e_1, \cdots, e_{n-1}, e_n, e_n^{-1}].$$

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Theorem (Li-Nadler-Y., 2023, simplified version)

Assume G is simply-connected (or more generally if the derived group of G is simply-connected), then restriction to $T \times T$ gives an isomorphism of $\mathbb{C}$-algebras

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Theorem holds over $\mathbb{Q}$ for split simply-connected groups.

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What happens if G is not simply-connected?

Example: $G = \mathrm{PGL}_2$. Consider the following pair

$$g_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \in \mathrm{PGL}_2.$$

They satisfy $g_1 g_2 g_1^{-1} g_2^{-1} = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} = 1 \in \mathrm{PGL}_2$.

However, (g_1, g_2) cannot be simultaneously diagonalized, or even simultaneously put in upper triangular form (check this!)

Those pairs that can be simultaneously upper triangularized form a closed G-invariant subscheme of $\mathcal{C}_G^2$; (g_1, g_2) above lies in the complement. $\Rightarrow$ restriction to $T \times T$ is not an isomorphism.

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Assume G is semisimple. Let $G^{\mathrm{sc}} \to G$ be the universal cover, whose kernel is $\pi_1(G)$. For $(g_1, g_2) \in \mathcal{C}_G^2$, take arbitrary liftings $\widetilde{g}_1, \widetilde{g}_2 \in G^{\mathrm{sc}}$, then consider $c = \widetilde{g}_1 \widetilde{g}_2 \widetilde{g}_1^{-1} \widetilde{g}_2^{-1} \in \pi_1(G)$. This is independent of the choice of liftings.

The assignment $(g_1, g_2) \mapsto c \in \pi_1(G)$ is a discrete invariant, and gives a decomposition

$$\mathcal{C}_G^2 = \coprod_{c \in \pi_1(G)} \mathcal{C}_G^2(c). \tag{0.1}$$

For general reductive group G, change $\pi_1(G)$ to $\pi_1(G^{\mathrm{der}})$. So that

$$\mathcal{O}(\mathcal{C}_G^2)^G = \prod_{c \in \pi_1(G^{\mathrm{der}})} \mathcal{O}(\mathcal{C}_G^2(c))^G.$$

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General statement

For each $c \in \pi_1(G^{\mathrm{der}})$, Borel–Friedman–Morgan defined a Levi subgroup $L_c \subset G$ (up to conjugacy, smallest Levi that contains a pair in $\mathcal{C}^2_G(c)$). Let T_c be the abelianization of L_c , and W_c the Weyl group $N_G(L_c)/L_c$ of L_c.

Theorem (Li-Nadler-Y., 2023)

Let G be a connected reductive group, and $c \in \pi_1(G^{\mathrm{der}})$. Then there is a canonical isomorphism of $\mathbb{C}$-algebras

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Variants

Notation:

- $\mathfrak{g}$: Lie algebra of G.
- $\mathcal{C}_{G,\mathfrak{g}}$: the scheme of pairs $(g, X) \in G \times \mathfrak{g}$ such that $\mathrm{Ad}(g)X = X$.
- $\mathcal{C}^2_{\mathfrak{g}}$: the scheme of pairs $(X_1, X_2) \in \mathfrak{g} \times \mathfrak{g}$ such that $[X_1, X_2] = 0$.

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Let G be a connected reductive group. The restriction maps give isomorphisms of $\mathbb{C}$-algebras

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Historical Remarks

- All these statements were known up to nilpotent elements: work of Joseph, Smilga-Kac, Borel–Friedman–Morgan. However, the reducedness question of $\mathcal{O}(\mathcal{C}_G^2)^G$ has been open for many years. So our essential contribution is showing that these rings of invariant functions are **reduced**.
- For commuting d-tuples $\mathcal{C}_{\mathfrak{g}}^d$ of Lie algebra $\mathfrak{g}$ and arbitrary d, similar result is known for **classical groups**: Gan–Ginzburg, Domokos, Vaccarino ($\mathfrak{gl}_n$); T.H. Chen–B.C.Ngô ($\mathfrak{sp}_{2n}$); L.Song–X.Xia –J.Xu ($\mathfrak{o}_n$).
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Derived version

The equations $g_1 g_2 = g_2 g_1$ are not all independent (not a complete intersection): if they were, the commuting scheme would have dimension equal to $\dim G$; however, it has dimension $\dim G + \dim T$.

The scheme $\mathcal{C}^2_G$ has a derived version $\mathfrak{C}^2_G$, taking into account of redundant defining equations. It fits into a derived Cartesian square

$$\begin{array}{ccc} \mathfrak{C}^2_G & \longrightarrow & G \times G \\ \downarrow & & \downarrow{\scriptstyle (g_1, g_2 \mapsto g_1 g_2 g_1^{-1} g_2^{-1})} \\ \{1\} & \hookrightarrow & G \end{array}$$

Its ring of functions is a differential graded algebra (in cohomological degrees ≤ 0):

$$\mathcal{O}(\mathfrak{C}^2_G) = \mathcal{O}(G \times G) \overset{\mathbf{L}}{\otimes}_{\mathcal{O}(G)} \mathbb{C}$$

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$$\mathcal{O}(\mathfrak{C}^2_G(c))^G \xrightarrow{\sim} \left(\mathcal{O}(T_c \times T_c) \otimes \wedge(\mathfrak{t}_c^*)\right)^{W_c}.$$

Here $\mathfrak{t}_c = \mathrm{Lie}(T_c)$, and $\wedge(\mathfrak{t}_c^*)$ has generators in degree -1.
Similar statements for the derived versions of $\mathcal{C}^{G,\mathfrak{g}}$ and $\mathcal{C}_{\mathfrak{g}}$ hold (by inserting $\wedge(\mathfrak{t}_c^*)$).

When G is simply-connected, this was conjectured by Berest–Ramadoss–Yeung (2017).

Proof outline

A geometric situation where commuting schemes naturally appear: consider G-local systems on a two-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. Monodromy operators along the meridian and longitude of $\mathbb{T}^2$ give two commuting elements $g_1, g_2 \in G$.

The (derived) moduli stack of G-local systems on $\mathbb{T}^2$ is

$$\mathrm{Loc}_G(\mathbb{T}^2) = \mathfrak{C}^2_G/_{\mathrm{Ad}}G.$$

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Betti geometric Langlands

Conjectured by Ben-Zvi–Nadler. Variant of (de Rham) geometric Langlands conjecture (Beilinson–Drinfeld, Arinkin–Gaitsgory).

- From now on, previous G will be denoted by $G^\vee$.
- G: connected reductive group over $\mathbb{C}$ Langlands dual to $G^\vee$.
- Example: $G^\vee = \mathsf{Sp}_{2n}$, $G = \mathsf{SO}_{2n+1}$.
- X: compact connected Riemann surface.
- $\mathsf{Bun}_G(X)$: moduli stack of principal G-bundles on X (alg. curve).
- $\mathsf{Loc}_{G^\vee}(\underline{X})$: moduli stack of $G^\vee$-local system on the underlying topological surface $\underline{X}$.

Conjecture (Betti geometric Langlands)

There is an equivalence of dg-categories

$$D_{\mathcal{N}}(\mathsf{Bun}_G(X), \mathbb{C}) \cong \mathsf{IndCoh}_{\mathcal{N}^\vee}(\mathsf{Loc}_{G^\vee}(\underline{X})).$$

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Genus one

Now suppose X has genus 1. Then $\underline{X} = \mathbb{T}^2$. Recall

$$\mathrm{Loc}_{G^\vee}(\underline{X}) \cong \mathfrak{C}^2_{G^\vee}/G^\vee.$$

Therefore, we have

$$\mathcal{O}(\mathfrak{C}^2_{G^\vee})^{G^\vee} \cong \mathbf{R}\mathrm{End}_{\mathrm{Loc}_{G^\vee}(\underline{X})}(\mathcal{O}).$$

If we assume Betti geometric Langlands, then $\mathcal{O} \in \mathrm{Coh}(\mathrm{Loc}_{G^\vee}(\underline{X}))$ corresponds to an object $\mathcal{W} \in D(\mathrm{Bun}_G(X))$, and

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Our proof morally follow this strategy, except that we work with another model for $D(\mathrm{Bun}_G(X))$.

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Affine Hecke category

- Finite Hecke category: $\mathcal{H}_G = D_{mon}(N\backslash G/N)$. Here mon means locally constant along left and right T-orbits. This is a monoidal dg-category.
- Affine Hecke category: $\mathcal{H}_{LG} = D_{mon}(\mathbf{I}^+\backslash LG/\mathbf{I}^+)$. Here $LG = G(\mathbb{C}((t)))$ is the loop group of G, and $\mathbf{I}^+ \subset G(\mathbb{C}[\![t]\!])$ is the preimage of N under evaluation $t \mapsto 0$. Also a monoidal dg-category.
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- Monoidal equivalence $\mathcal{H}_{LG} \cong \mathsf{IndCoh}(St_{G^\vee}/G^\vee)$. Mild generalization of Bezrukavnikov's Theorem.
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- Known: $hh(\mathcal{H}_{LG}) \cong \mathsf{IndCoh}_{\mathcal{N}^\vee}(\mathfrak{C}^2_{G^\vee}/G^\vee)$.
- Let $\mathcal{W} \leftrightarrow \mathcal{O}$ under this equivalence.
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More details on $\mathcal{H}_{LG}$

Notation:

- Standard parahoric subgroups $\mathbf{P}_J \subset LG$ indexed by certain subsets J of affine simple roots of LG.
- Each $\mathbf{P}_J$ has a Levi quotient L_J, a connected reductive group.
- The finite Hecke category $\mathcal{H}_J$ of L_J is a full subcategory of $\mathcal{H}_{LG}$.

Theorem (J.Tao-Travkin)

Assume G is simply-connected. Then the natural functor

$$\mathrm{colim}_J^{\otimes} \mathcal{H}_J \to \mathcal{H}_{LG}$$

is an equivalence of monoidal categories. (partially order the J's by inclusion)

More details on $hh(\mathcal{H}_{LG})_0$

Theorem (Li-Nadler-Y.)

Assume G is simply-connected. Then the natural functor

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is a full embedding. We define $hh(\mathcal{H}_{LG})_0$ to be the image of this embedding.

More precisely, $hh(\mathcal{H}_{LG})$ has a filtration by full subcategories $hh(\mathcal{H}_{LG})_{\leq\nu}$ indexed by **Newton points** ν, starting with $hh(\mathcal{H}_{LG})_0$. They form a **recollement structure** on $hh(\mathcal{H}_{LG})$, similarly to sheaves on stratified spaces.

Proof ingredients: Tao-Travkin theorem; parabolic character sheaves (Lusztig); Results of He–Nie and Xuhua He; categorical contraction (Morse) principle .

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Character sheaves

How to calculate $hh(\mathcal{H}_J)$?

Character sheaves on a reductive group H (Lusztig, reformualted by Mirkovic–Vilonen): $CS(H) \subset D(H/_{\mathrm{Ad}}H)$ is the full subcategory of sheaves with nilpotent singular support (controlled jumps).

Theorem

Let H be a connected reductive group over $\mathbb{C}$. Then there is a natural equivalence

$$hh(\mathcal{H}_H) \cong \mathcal{CS}(H).$$

Variants of this theorem appeared in work of Ben-Zvi–Nadler, Bezrukavnikov–Finkelberg–Ostrik and Lusztig.

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$\mathcal{W}$ and its endomorphism ring

- $\mathcal{W}$: image of the **Whittaker object** in $\mathcal{CS}(G)$.
- Penghui Li: combinatorial description of $\mathrm{colim}_J \mathcal{CS}(L_J)$ in terms of double affine Weyl groups.
- How to see decomposition of $\mathbf{R}\mathrm{End}(\mathcal{W})$ by $c \in \pi_1(G^{\vee,\mathrm{der}})$?
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Toy case: $G = T$ is a torus

- Let $\Lambda = \mathbb{X}_*(T)$.
- $hh(\mathcal{H}_T)_0 = \mathcal{CS}(T)_\Lambda$, where Λ acts trivially on T.
- $\mathcal{CS}(T) = D_{\text{loc.const}}(T/_{\text{Ad}}T) \cong \text{Loc}(T) \otimes D(\text{pt}/T) \cong \mathbb{C}[\Lambda] \otimes H_*(T)\text{-mod} \cong \mathcal{O}(T^\vee) \otimes \wedge(\mathfrak{t}[1])\text{-mod}$.
- $\mathcal{CS}(T)_\Lambda \cong \mathbb{C}[\Lambda]\text{-mod}(\mathcal{CS}(T)) \cong \mathcal{O}(T^\vee \times T^\vee) \otimes \wedge(\mathfrak{t}[1])\text{-mod}$
- $\mathcal{W}$ corresponds to the free module of rank one. Therefore,

$$\mathbf{R}\text{End}(\mathcal{W}) \cong \mathcal{O}(T^\vee \times T^\vee) \otimes \wedge(\mathfrak{t}[1]).$$

Cocenter conjecture

Conjecture (Ben-Zvi–Nadler)

Let X be a genus one Riemann surface. There is an equivalence

$$D_{\mathcal{N}}(\mathrm{Bun}_G(X)) \cong hh(\mathcal{H}_{LG}).$$

- This is a **Langlands functoriality** or **base change** type statement: purely in terms of automorphic sides.
- This is a combination of Betti geometric Langlands conjecture in genus 1, Bezrukavnikov's theorem and Ben-Zvi–Nadler–Preygel's theorem.
- The part $hh(\mathcal{H}_{LG})_0 \subset hh(\mathcal{H}_{LG})$ corresponds to sheaves on $\mathrm{Bun}_G(X)$ supported on the open substack of semistable bundles. The subcategories $hh(\mathcal{H}_{LG})_{\leq\nu}$ corresponds to the Harder-Narasimhan stratification of $\mathrm{Bun}_G(X)$.

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Happy birthday SMS!