Small scale formations in fluid equations with gravity

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Incompressible Porous Media (IPM) equation

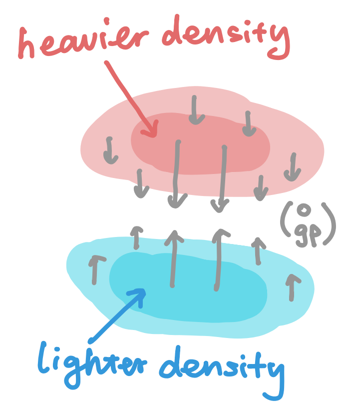
- $\rho(x,t)$: density of incompressible fluid in porous media.
- $\boldsymbol{u}(x,t)$: velocity field of fluid.

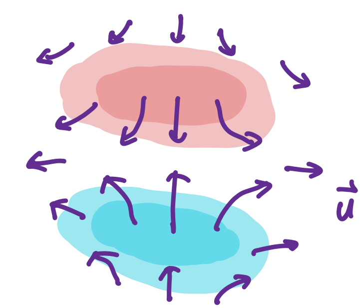
$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \nabla \cdot \boldsymbol{u} = 0 \end{cases} \quad \text{in } \Omega \times [0,T).$$

Here the spatial domain Ω is $\mathbb{R}^2$, $\mathbb{T}^2$, or $S = \mathbb{T} \times [-\pi, \pi]$.

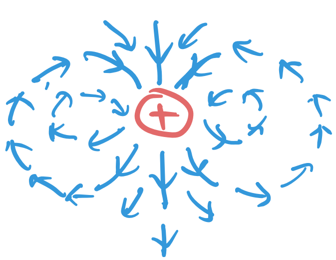
- Darcy's law for flow in porous media:

$$\boldsymbol{u} = -\nabla p - \begin{pmatrix} 0 \\ g\rho \end{pmatrix}.$$





- Setting $g = 1$, the Biot-Savart law becomes

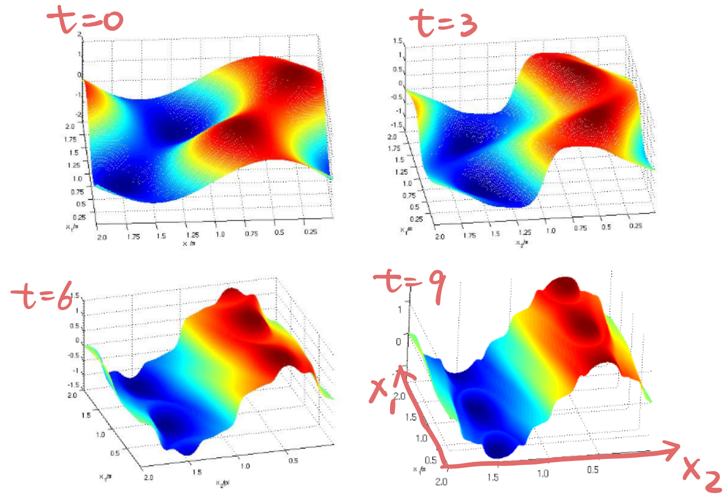


$$\boldsymbol{u} = \partial_{x_1} \nabla^{\perp} (-\Delta_{\Omega})^{-1} \rho.$$

On well-posedness of IPM

- Note that IPM closely resembles 2D Euler equation $\omega_t + u \cdot \nabla\omega = 0$, except that $u = \nabla^\perp(-\Delta)^{-1}\omega$ in 2D Euler, whereas $u = \partial_{x_1}\nabla^\perp(-\Delta)^{-1}\rho$ in IPM.

- Córdoba–Gancedo–Orive '07: Local well-posedness in H^s, and various blow-up criteria. Numerics suggest that $\|\nabla\rho\|_{L^\infty}$ is growing as $t \to \infty$, although no evidence for finite-time blow-up.



- Elgindi '14 and Castro–Córdoba–Lear '19: Global WP and convergence when ρ_0 is close to the stable stratified state $\rho = -x_2$.
- But for general smooth initial data, it is still an open question whether solutions are globally well-posed in time.

Small scale formation of IPM in $\mathbb{R}^2$

Goal: Assuming a global-in-time solution ρ in $\Omega \times [0, \infty)$, we want to rigorously prove the growth of $\nabla \rho$ as $t \to \infty$ for some initial data.

Theorem (Kiselev–Y. '21)

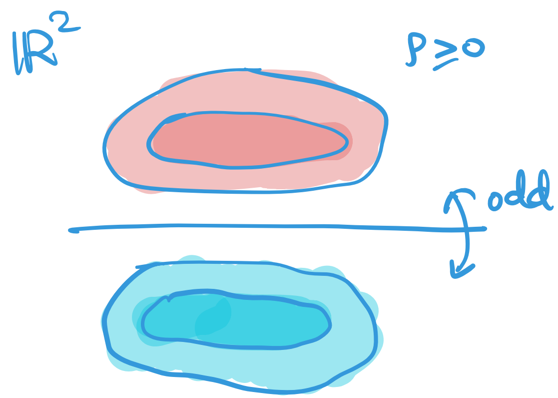
Assume $\rho_0 \in C_c^\infty(\mathbb{R}^2)$ is odd in x_2, and $\rho_0 \geq 0$ in $\mathbb{R} \times \mathbb{R}^+$. If the solution remains smooth for all time, it satisfies

$$\int_0^\infty \|\rho(t)\|_{\dot{H}^s(\mathbb{R}^2)}^{-\frac{4}{s}} dt \leq C(s, \rho_0) < \infty \qquad \textit{for all } s > 0.$$

- It implies $\limsup_{t\to\infty} t^{-\frac{s}{4}} \|\rho(t)\|_{\dot{H}^s(\mathbb{R}^2)} = \infty$ for all $s > 0$, thus $\rho(t)$ has infinite-in-time growth in $\dot{H}^s$ norm for any $s > 0$.
- Here the $s > 0$ range is sharp, since $\|\rho(t)\|_{L^2}$ is invariant in time.

Sketch of the proof: problem set-up

- Set up of initial data:



  (Note that the odd symmetry is preserved for all times.)

- Main tool: monotonicity of the potential energy

$$E(t) := \int_{\mathbb{R}^2} \rho(x,t) x_2 \, dx.$$

- A quick computation gives (using $\boldsymbol{u} = \partial_{x_1} \nabla^{\perp} (-\Delta)^{-1} \rho$)

$$\frac{d}{dt} E(t) = \int_{\mathbb{R}^2} \rho u_2 \, dx = \int_{\mathbb{R}^2} \rho \, \partial^2_{x_1 x_1} (-\Delta)^{-1} \rho \, dx = -\|\partial_{x_1} \rho\|^2_{\dot{H}^{-1}} =: -\delta(t).$$

- $\rho(\cdot, t) \geq 0$ in the upper half plane $\Longrightarrow$ as long as we have a smooth solution,

$$E(t) \geq 0 \qquad \text{for all } t \geq 0.$$

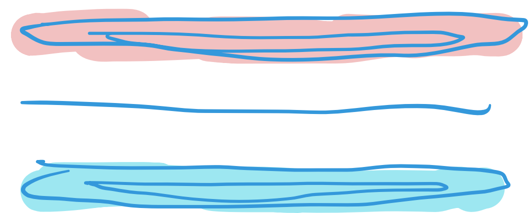
Relating $\delta(t)$ with $\|\rho\|_{\dot{H}^s}$

- Recall: $\delta(t) := \|\partial_{x_1}\rho\|^2_{\dot{H}^{-1}}$ satisfies $\int_0^\infty \delta(t) \le E(0) < \infty$.

- Note that $\delta(t) = 0 \iff \partial_{x_1}\rho(t) \equiv 0$.
  So we expect
  "$\delta(t) \ll 1 \implies \|\rho(t)\|_{\dot{H}^s} \gg 1$".

$\delta(t) \ll 1$

$\Downarrow$



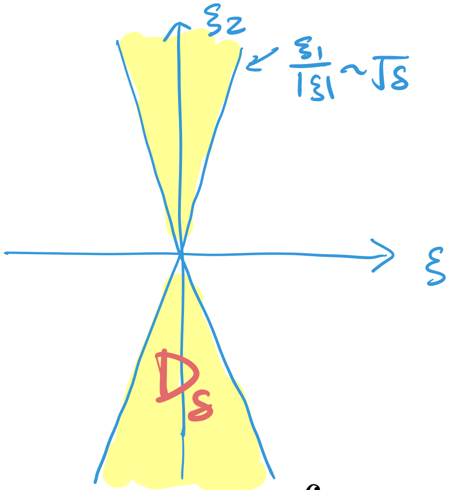
- Goal: $\|\rho(t)\|_{\dot{H}^s} \gtrsim \delta(t)^{-s/4}$ for all $s > 0$.

  Plugging it into $\int_0^\infty \delta(t)dt < C$ finishes the proof!

- Idea of proof: On the Fourier side,

  $\int_{\mathbb{R}^2} |\hat{\rho}(\xi)|^2 d\xi = \|\rho_0\|^2_{L^2}$ is conserved;

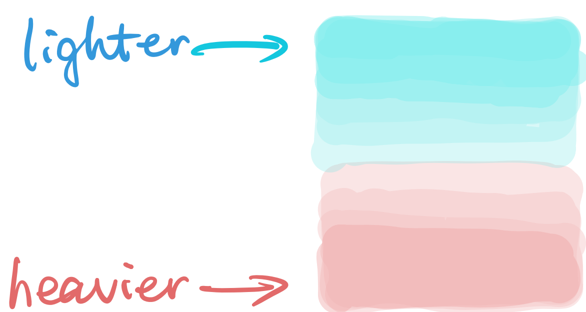
  $|\hat{\rho}(\xi)| \le \|\rho_0\|_{L^1}$ is bounded.



So $\delta(t) = \displaystyle\int_{\mathbb{R}^2} \frac{\xi_1^2}{|\xi|^2} |\hat{\rho}(\xi)|^2 d\xi \ll 1 \implies \|\rho(t)\|^2_{\dot{H}^s} \ge \int_{\mathbb{R}^2} \xi_2^{2s} |\hat{\rho}(\xi)|^2 d\xi \gtrsim \delta^{-s/2}$.

Stability v.s. instability of stratified states

- Note that for the IPM, any horizontal stratified state $\rho_s(x) = g(x_2)$ is stationary. Is it stable or not?
- $\eta := \rho - \rho_s$ satisfies $\eta_t + u \cdot \nabla \eta = -g'(x_2)u_2$ with $u = \nabla^\perp(-\Delta)^{-1}\partial_{x_1}\eta$.
- Linearized equation: $\eta_t = -g'(x_2)\overbrace{(-\Delta)^{-1}\partial_{x_1}^2}^{\text{negative operator}}\eta$.
- For $g(x_2) = -x_2$, asymptotic stability for the nonlinear equation was established by Elgindi '14 in $\mathbb{R}^2$ for H^{20} and above, and Castro–Córdoba–Lear '18 in the strip $S = \mathbb{T} \times [0, 1]$ for H^{10} and above.



$\rho_s = -x_2$
(nonlinearly stable in H^s for large s!)

- Interestingly, we'll show this linearly stable steady state in a strip is nonlinearly unstable in H^s if s is low!

Nonlinear instability of stratified states in a strip

We prove nonlinear instability for *any* stratified states in a strip, including the nonlinearly stable ones (in H^{10} or above) $\rho_s = -x_2$:

Theorem (Kiselev–Y. '21)

Let $\rho_s \in C^\infty(S)$ be any stationary solution. For any $\epsilon, \gamma > 0$, there exists an initial data $\rho_0 \in C^\infty(S)$ satisfying

$$\|\rho_0 - \rho_s\|_{H^{2-\gamma}(S)} \le \epsilon,$$

such that the solution satisfies (if it remains smooth for all times)

$$\limsup_{t\to\infty} t^{-\frac{s}{2}} \|\rho(t) - \rho_s\|_{\dot{H}^{s+1}(S)} = \infty \quad \text{for all } s > 0.$$

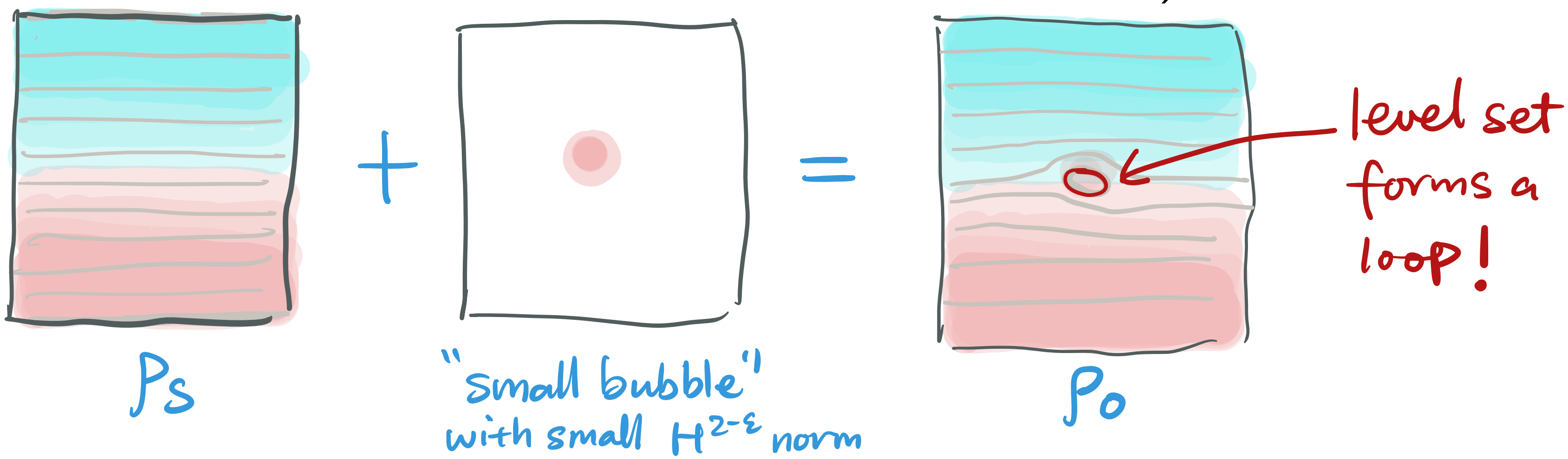
Combining the stability and instability results together, we know in a strip S, the steady state $\rho_s = -x_2$ is

- stable in H^m for $m \ge 10$ (Castro–Córdoba–Lear '18)
- unstable in H^m for $1 < m < 2$ (Kiselev-Y. '21)

Such phenomenon is common in the study of PDEs: the stability/instability of steady states often depends on the norm used.

Proof: adding a small "bubble"

- **Idea:** add a little "bubble" locally to create two closed level sets in ρ_0. (Its $H^{2-\epsilon}$ norm can be made small, but not H^2 and above.)



- The closed loops remain closed during the evolution, meaning $\rho(t)$ can never get too close to a perfect stratified state – can show that

$$\int_S |\partial_{x_1}\rho(x,t)|dx > c(\rho_0) > 0 \text{ for all } t.$$

- Combining this with $\delta(t) = \|\partial_{x_1}\rho\|^2_{\dot{H}^{-1}}$ being integrable in time immediately leads to infinite-in-time growth of $\|\partial_{x_1}\rho\|^2_{\dot{H}^s}$ for $s > 0$.

2D viscous Boussinesq equation without density diffusivity

- 2D viscous Boussinesq equation in $\mathbb{T}^2$ with no density diffusivity:

$$\begin{cases} \rho_t + u \cdot \nabla \rho = 0, \\ u_t + u \cdot \nabla u = -\nabla p - \rho e_2 + \Delta u, \\ \nabla \cdot u = 0, \end{cases}$$

- Global well-posedness in $H^{s-1} \times H^s$: Hou–Li '05, Chae '06, Larios–Lunasin–Titi '13, Hu–Kukavica–Ziane '13 & '15.
- Upper bound on $\|\rho(t)\|_{H^1}$: Ju '17 (double exp growth), Kukavica–Wang '19 (exp growth)
- But can $\|\rho\|_{H^1}$ grow to infinity as $t \to \infty$?

Theorem (Kiselev–Park–Y. '22, preprint)

There exists smooth initial data ρ_0, u_0 in $\mathbb{T}^2$ such that the global-in-time smooth solution (ρ, u) satisfies $\limsup_{t\to\infty} t^{-1/6}\|\rho(t)\|_{H^1} = \infty$.

- The proof has a similar flavor as the IPM case, but it's more delicate since the potential energy is not monotone for Boussinesq.

Inviscid 2D Boussinesq equation

- In the inviscid case, let us work with the variables ρ and vorticity ω:

$$\begin{cases} \rho_t + u \cdot \nabla \rho = 0, \\ \omega_t + u \cdot \nabla \omega = -\partial_1 \rho, \end{cases}$$

  where u can be recovered from the Biot-Savart law $u = \nabla^\perp (-\Delta)^{-1} \omega$.
- Whether smooth initial data can lead to a blow-up in $\mathbb{T}^2$ or $\mathbb{R}^2$ is an outstanding open question.
- It is well-known that away from the axis of symmetry, the 3D axisymmetric Euler equation is closely related to 2D Boussinesq:

$$\begin{cases} D_t(ru^\theta) = 0, \\ D_t\left(\frac{\omega^\theta}{r}\right) = r^{-4}\partial_z(ru^\theta)^2, \end{cases}$$

  where $D_t := \partial_t + u^r\partial_r + u^z\partial_z$ is the material derivative, and (u^r, u^z) can be recovered from ω^θ / r by a similar Biot-Savart law.

Blow-up for inviscid 2D Boussinesq and 3D Euler

In the presence of boundary, or for non-smooth initial data, there are many exciting developments on finite-time blow-up:

- Luo–Hou '14: convincing numerical evidence for blow-up at the boundary for 3D axisymmetric Euler
- Elgindi–Jeong '20: blow-up in domain with a corner
- Elgindi '21: blow-up for $C^{1,\alpha}$ solutions for 3D Euler
- Chen–Hou '21: blow-up for $C^{1,\alpha}$ solutions with boundary
- Wang–Lai–Gómez-Serrano–Buckmaster '22: numerics for approximate self-similar blow-up solution using physics-informed neural networks.
- Chen–Hou '22: stable nearly self-similar blowup for smooth solutions (combination of analysis + computer-assisted estimates)

Question: Can one construct solutions with infinite-in-time growth for more general class of initial data?

Infinite-in-time growth in a strip

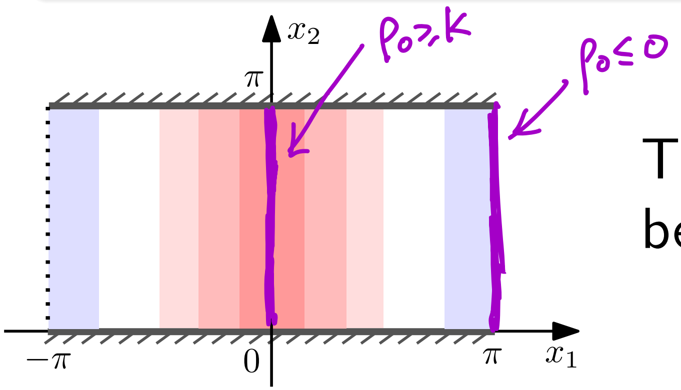
Theorem (Kiselev–Park–Y. '22, preprint)

Let $\Omega = \mathbb{T} \times [0,\pi]$. Let $\rho_0 \in C^\infty(\Omega)$ be even in x_1, and $\omega_0 \in C^\infty(\Omega)$ be odd in x_1, with $\int_{[0,\pi]\times[0,\pi]} \omega_0 dx \geq 0$. Assume that there exists $k_0 > 0$ such that $\rho_0 \geq k_0 > 0$ on $\{0\} \times [0,\pi]$, and $\rho_0 \leq 0$ on $\{\pi\} \times [0,\pi]$. Then the solution satisfies the following during its lifespan:

$$\|\omega(t)\|_{L^p(\Omega)} \gtrsim t^{3-\frac{2}{p}},$$

$$\|u(t)\|_{L^\infty(\Omega)} \gtrsim t,$$

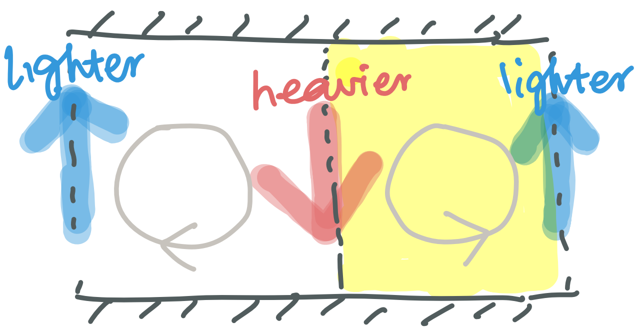
$$\sup_{\tau\in[0,t]} \|\nabla\rho(\tau)\|_{L^\infty(\Omega)} \gtrsim t^2.$$



The proof is a soft argument, based on an interplay between various monotone and conservative quantities.

Monotonicity of vorticity integral

- Let Q be the right half of the strip. Simple but useful observation:



$$\frac{d}{dt}\int_Q \omega dx = \cancel{\int_Q -u\cdot\nabla\omega dx}^{\,0} - \int_Q \partial_1\rho dx$$
$$= \int_0^\pi \underbrace{\rho(0,x_2,t)}_{\geq k_0} dx_2 - \int_0^\pi \underbrace{\rho(\pi,x_2,t)}_{\leq 0} dx_2$$
$$\geq k_0\pi.$$

- Since $\int_{\partial Q} u\cdot dl = \int_Q \omega dx \geq k_0\pi t$, we have $\|u(t)\|_{L^\infty}$ grows at least linearly.
- On the other hand, $\|u\|_{L^2}$ is bounded for all times by energy conservation.
- Combining the boundedness of $\|u\|_{L^2(Q)}$ and linear growth of $\int_{\partial Q} u\cdot dl$, we know u must change rapidly in a small neighborhood of ∂Q, leading to super-linear growth of ∇u (and ω).

3D axisymmetric Euler in an annular cylinder

Using a similar idea, we obtain infinite-in-time growth for the 3D axisymmetric Euler equation in an annular cylinder

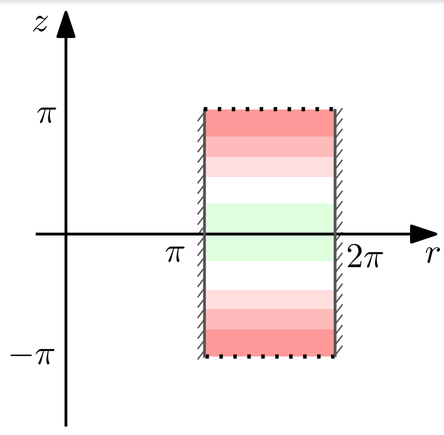
$$\Omega = \{(r,\theta,z) : r \in [\pi, 2\pi]; \theta \in \mathbb{T}, z \in \mathbb{T}\}.$$

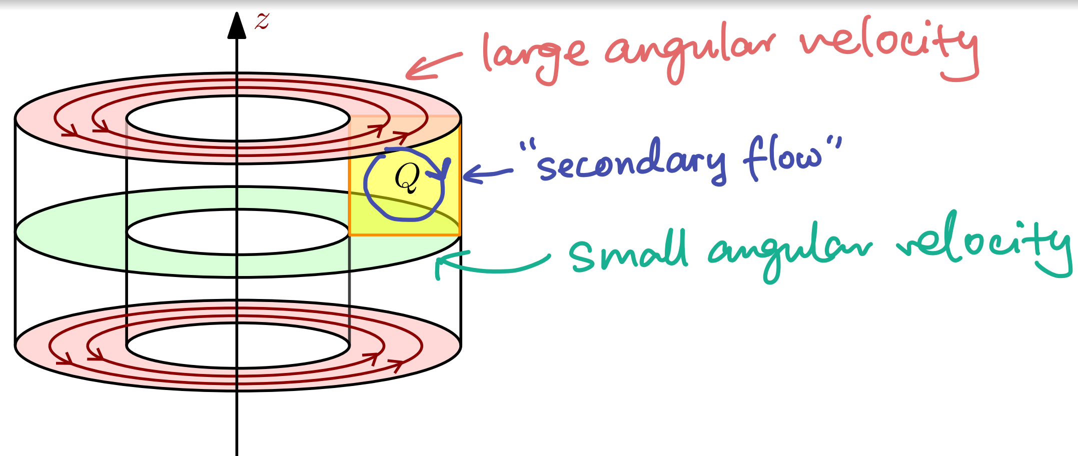
Theorem (Kiselev–Park–Y. '22, preprint)

Let $u_0^\theta \in C^\infty(\Omega)$ be even in z, $\omega_0^\theta \in C^\infty(\Omega)$ odd in z, with $\int_0^\pi \int_\pi^{2\pi} \omega_0^\theta drdz \geq 0$. Assume there exists $k_0 > 0$ such that $u_0^\theta \geq k_0 > 0$ on $z = \pi$, and $|u_0^\theta| \leq \frac{1}{8}k_0$ on $z = 0$. Then the solution to axisymmetric 3D Euler satisfies

$$\|\omega^\theta(t)\|_{L^p(\Omega)} \gtrsim t^{3-\frac{2}{p}} \quad \text{and} \quad \|u(t)\|_{L^\infty(\Omega)} \gtrsim t$$

during the lifespan of the solution.





Thank you for your attention!