Kähler-Einstein metric, K-stability and moduli spaces

PKU Mathematics Forum

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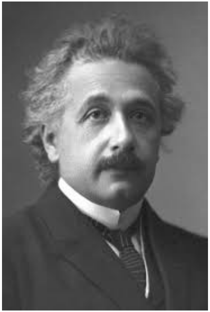
- Part 1: Kähler-Einstein Problem of Fano varieties
- Part 2: Moduli of Fano varieties
- Part 3: Higher rank finite generation

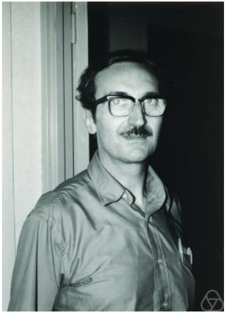
Part 1: Kähler-Einstein Problem of Fano varieties

- X is a complex manifold. Let g be a Hermitian metric on X.
- (Kähler 1933) Kähler metric: g can be locally written as $\frac{\partial^2 f}{\partial z_\alpha \partial \bar{z}_\beta}\, dz_\alpha \otimes\, d\bar{z}_\beta$. This is true if and only if the associated form ω_g satisfies $d\omega_g = 0$.
- $\partial\bar{\partial}$-Lemma: two Kähler forms ω_1 and ω_2 are in the same class, if and only if $\omega_1 - \omega_2 = i\partial\bar{\partial}\varphi$ for some φ.

Kähler-Einstein metrics

- Kähler-Einstein problem (Kähler 1933, Calabi 1950s): find a Kähler metric ω such that $\mathrm{Ric}(\omega) = \lambda\omega$ for a constant λ.
- $[\mathrm{Ric}(\omega)] = c_1(X)$.
- $\dim = 1$, Poincaré Uniformization Theorem.
- $\lambda = 0$ or -1, solved by Yau and Aubin/Yau in 70s.
- (Matsushima 57) $\lambda = 1$, i.e. X is Fano, there is an obstruction: X has KE implies $\mathrm{Aut}(X)$ is reductive.
- Program: Characterize when $\mathrm{Ric}(\omega) = \omega$ has a solution.
- More Ambitious Program: parametrizing them using good moduli spaces.





Variational method

- Fix ω_0 with $[\omega_0] = c_1(X)$, write $\omega - \omega_0 = i\partial\bar{\partial}\varphi$ and $\omega_0 - \mathrm{Ric}(\omega_0) = i\partial\bar{\partial}F$.
- Then $\mathrm{Ric}(\omega) = \omega$ is equivalent to the complex Monge-Ampere equation
$$(\omega_0 + i\partial\bar{\partial}\varphi)^n = e^{F-\lambda\varphi}\omega_0^n.$$
- Let $\mathcal{H} = \{\varphi \big| \omega_0 + i\partial\bar{\partial}\varphi > 0\}$. Mabuchi(86): there is a K-energy functional (Mabuchi functional)
$$M \colon \mathcal{H} \to \mathbb{R},$$
such that a critical point of M precisely corresponds to the solution of the complex Monge-Ampere equation, i.e. a Kähler-Einstein metric.
- Ding (88): Ding functional $D \colon \mathcal{H} \to \mathbb{R}$ with the same property.

Finite dimensional toy model: GIT

- A reductive group $G = K^{\mathbb{C}}$ acts on a polarized manifold (X, L).
- Let $\|\cdot\|$ be a K-invariant norm on L.
- For $x \in X$, we define the function

$$f\colon G/K \to \mathbb{R}, g \to \log\|g \cdot \hat{x}\|,$$

  where $\hat{x} \in L_x$ is a non-zero lift of x.
- $\xi\colon \mathbb{C}^* \to G/K$ gives geodesic. $\lim_{t\to\infty} f'(e^{it\xi} \cdot x) = w_\xi$: the weight of the $\mathbb{C}^*$-action on L_{x_0} where $x_0 = \lim_{\lambda\to 0} \lambda \cdot x$.
- f has a unique minimum

$$\iff \lim_{t\to\infty} f'(e^{it\xi} \cdot x) > 0 \text{ for any } \xi \in \mathrm{Lie}(K)_{\mathbb{R}}$$

$$\iff w_\xi > 0 \text{ for all } \xi \in \mathrm{Lie}(K)_{\mathbb{Q}} \text{ (Kempf-Ness)}$$

$$\iff x \text{ is GIT stable (Hilbert-Mumford)}$$

Yau-Tian-Donaldson Conjecture

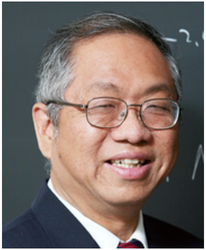
- (Yau 80s) The existence of a KE metric on a Fano manifold should relate to 'algebraic stability' theory.
- (Ding-Tian 92, Tian 97) Consider $\mathcal{X} \to \mathbb{A}^1$ a $\mathbb{C}^*$-equivariant degeneration $\mathcal{X} \subset \mathbb{P}^N$ of $X \xrightarrow{|-rK_X|} \mathbb{P}^N$ by $\mathbb{C}^* \to \mathrm{PGL}(N+1)$. X has KE implies $\mathrm{Fut}(\mathcal{X}) \geq 0$.
- Tian (97) defined K-(semi,poly)stability notions, by looking at the sign of $\mathrm{Fut}(\mathcal{X})$ for all $\mathcal{X}$ (for all r).
- (Donaldson 02) Reformulate Futaki invariants in algebraic terms.

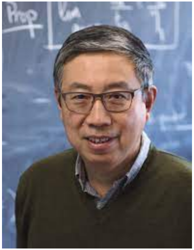
Theorem (YTD Conjecture)

For a Fano variety, it has a KE metric if and only if it is K-(poly)stable.

Remark

We will see, the algebraic part of solving this problem is contained in the bigger program of constructing moduli spaces.





Easy direction and Ding stability

Theorem (Tian 97, Berman 12)

The existence of a unique KE metric implies K-stability.

- $\sigma_t : t \to e^{-t} \in \mathbb{C}^*$ and $\lim_{t\to+\infty} \frac{\mathrm{d}M(\frac{1}{r}\sigma_t^*\omega_{\mathrm{FS}})}{\mathrm{d}t} = \mathrm{Fut}(\mathcal{X})$.
- Berman introduced the notion of Ding stability.
- Ding stability fits better into higher dimensional geometry.
- The algebraic foundation: transfers from GIT to minimal model program (MMP).
- (Fujita) Ding stability is equivalent to K-stability, following from Li-X's specialization theory.

Characterizing K-stability using valuations

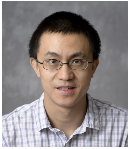
- Let $A_X(E) := \mathrm{mult}_E(K_{Y/X}) + 1$ be the log discrepancy of E on a birational model $\mu\colon Y \to X$. X is Kawamata log terminal (klt) if $A_X(E) > 0$ for all E.
- $S_X(E)$ is the expected vanishing order, i.e.,

$$S_X(E) = \frac{1}{(-K_X)^n} \int_0^\infty \mathrm{vol}(\mu^*(-K_X) - tE)dt.$$

- Set the stability function $\delta(X) = \inf_E \frac{A_X(E)}{S_X(E)}$. (If $\delta \le 1$, $\delta = \mathrm{Sup}\{t \mid \mathrm{Ric}(\omega) = t\omega + (1-t)\omega_0\}$)

Theorem (Fujita-Li Criterion)

Uniform K-stability $\iff \delta(X) > 1$; *K-stability* $\iff \frac{A_X(E)}{S_X(E)} > 1$ *for all* E.



Optimal destabilization

Theorem (Optimal Destablization, Liu-X.-Zhuang 21)

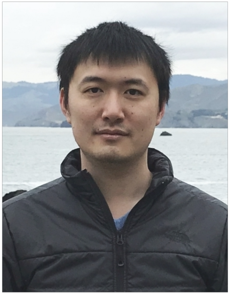
If $\delta(X) < \frac{n+1}{n}$, then there exists a divisor E, such that $\delta(X) = \frac{A_X(E)}{S_X(E)}$.

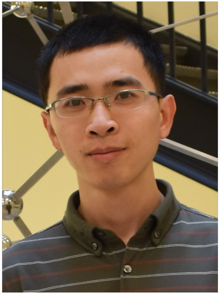
- This settles the compactedness of the moduli space of K-polystable Fano varieties.
- The case $\delta(X) = 1$ says (uniform K-stability) = (K-stability).
- The technical core is a higher rank finite generation theorem.

Theorem (Berman-Boucksom-Jonsson 15, Li-Tian-Wang 19, Li 19, Zhang 21)

For a general (possibly singular) Fano X, the uniform K-stability, implies the existence of a unique KE.

- Study the geometry of the space $\mathcal{E}^1$ of ω_0-psh functions with finite energy, which is a completion of $\mathcal{H}$.





Riemannian geometry approach

Theorem (Chen-Donaldson-Sun 12, Tian 12)

For a smooth Fano manifold, K-stability implies the existence of a KE.

- Continuity method: Fix a smooth $D \sim -mK_X$,

$$t_0 = \inf\left\{ t \ \mid \mathrm{Ric}(\omega) = (1-t)\omega + \frac{t}{m}[D] \text{ is solvable} \right\}.$$

- Compactedness says

$$t_i \searrow t_0, (X, \frac{t_i}{m}D, \omega_{i,KE}) \xrightarrow{\mathrm{GH}} (X_0, \frac{t_0}{m}D_0, \omega_{0,KE})$$

and $X \ncong X_0$. There is a $\mathbb{G}_m$-degeneration $\mathcal{X}$ of $X \rightsquigarrow X_0$ with $\mathrm{Fut}(\mathcal{X}) \leq 0$.

Part 2: Moduli of Fano varieties

Moduli of varieties: history

There has been a long history for people trying to parametrize varieties, going back to Abel, Jacobi, Riemann, Weierstrass, Teichmuller etc.

General moduli theory for $K_X > 0$:

- Curves of higher genus g ($g \geq 2$): $\mathcal{M}_g$, Mumford's geometric invariant theory (GIT); Deligne-Mumford compactification: $\overline{\mathcal{M}}_g$.
- Kollár-Shepherd-Barron (KSB) theory (88) proposes generalizing Deligne-Mumford construction to higher dimensions. By late 2010s, the program is completed, and there is a compact moduli parametrizing X with ample K_X.
- It is intertwining with the progress of the minimal model program theory (MMP).

Moduli of varieties: Fano

For Fano varieties, no algebraic geometers thought about constructing moduli for Fano varieties before 2010, since it seems impossible.....

- No natural extrinsic theory, in particular GIT does not fit.
- Li-X. (12): Families of Fano varieties: a family of Fano manifolds $X^\circ \to \Delta^\circ$, MMP often yields many possible Fano limits X_0, but no canonical choice.
- Solution: There is a stratification of the moduli stack of all Fano varieties. K-(semi)stable ones yields moduli.

K-moduli theorem

Theorem (K-moduli stack/space)

1. $\mathfrak{X}^{\mathrm{Kss}}_{n,V} = [Z/G]$ for a quasi-projective scheme Z and $G = \mathrm{PGL}(N+1)$ for some $N = N(n, V)$.
2. $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$ admits a separated good moduli space $X^{\mathrm{Kps}}_{n,V}$, whose points correspond to K-polystable Fano varieties.
   - (The $\mathbb{C}$-points $X^{\mathrm{Kps}}_{n,V}(\mathbb{C})$ precisely correspond to KE ones).
3. $X^{\mathrm{Kps}}_{n,V}$ is a proper.
4. the CM line bundle is ample on $X^{\mathrm{Kps}}_{n,V}$.

Good moduli space

- For a stack $\mathfrak{X}$, having a (separated) good moduli space X is a subtle property.
- (Étale) locally over X, $\mathfrak{X} \to X$ is covered by

$$[\mathrm{Spec}(A)/G] \to \mathrm{Spec}(A^{\mathrm{G}}),$$

  where G is a reductive group. We use a valuative criterion by Alper-Halpern-Leistner-Heinloth18 to check $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$.
- Study families of K-semistable Fano varieties over (equivariant) surfaces (Li-Wang-X.18, Blum-X.18, A-Blum-HL-X.19).
- Corollary: For a K-polystable Fano variety X, $\mathrm{Aut}(X)$ is reductive.

Some other progress

- Special theory: special degeneration (Li-X. 12) vs special valuations (Blum-Liu-X. 19).
- Positivity of CM line bundle: connecting stability of fibers with Harder-Narashimhan filtration of the base (Codogni-Patakfalvi 18, X-Zhuang 19).
- Local singularity theory: Li's normalized volume (15), Stable Degeneration Conjecture (X-Zhuang 22 etc.).
- Explicit verification: estimating $\delta(X)$ by the Abban-Zhuang method, moduli method (Liu and others).

Part 3: Higher rank finite generation

Higher rank finite generation

- Space of valuations $\mathrm{Val}(X)$: $v\colon K(X)\setminus\{0\}\to\mathbb{R}$, $v(\mathbb{C})=0$, $v(xy)=v(x)+v(y)$, $v(x+y)\geq\min\{v(x),v(y)\}$.
- The function

$$\delta : v \to A_X(v)/S_X(v)$$

can be defined for any nontrivial valuations $v\in\mathrm{Val}(X)$ with $A_X(v)<+\infty$.

Theorem (HRFG: δ-minimizer, Liu-X.-Zhuang 21)

Assume $\delta(X)<\frac{n+1}{n}$. Let v be a valuation which computes $\delta(X)$, then $\mathrm{gr}_v R$ is finitely generated for $R:=\bigoplus_m H^0(-mK_X)$, where $\mathrm{gr}_v R=\bigoplus_m\bigoplus_\lambda \mathcal{F}^\lambda R_m/\mathcal{F}^{>\lambda}R_m$, and

$$\mathcal{F}^\lambda R_m(\text{resp. } \mathcal{F}^{>\lambda}R_m)=\{s\in H^0(-mK_X)\mid v(s)\geq(\text{resp. } >)\lambda\}\,.$$

- The finite generation brings back the problem to a finite level. Once it is known, a small rational perturbation v' of v yields a divisorial minimizer of δ as predicted by the optimal destabilization theorem.

Minimizers on special model

- Special model (dlt Fano type model): $\mu\colon (Y, E = \sum_{i=1}^{r} E_i) \to X$ is a birational model, such that $-K_Y - E - \Delta$ is ample for some $\Delta \geq 0$, and (Y, E) is simple normal crossing over the generic point η of $\bigcap_{i=1}^{r} E_i$.

Theorem (Li-X.17, Blum-Liu-X. 19, Liu-X.-Zhuang 21, X-Zhuang 22)

Assume $\delta(X) < \frac{n+1}{n}$. A minimizer of δ exists and is monomial over $\eta \in (Y, E)$, where (Y, E) is a special model.

- One needs major boundedness results from birational geometry, e.g. Hacon-McKernan-X. 12, Birkar 16 etc..

Finite generation

Theorem (HRFG: birational version, X-Zhuang 22)

Let $\mu\colon (Y, E = \sum_{i=1}^{r} E_i) \to X$ *be a special model. Then for any valuation* v *monomial over* $\eta \in (Y, E)$, $\mathrm{gr}_v R$ *is finitely generated.*

- If v is a divisorial valuation ord_E, this directly follows from the minimal model program.
- For a higher rank valuation, this posts a substantial new challenge.

Remark

We also complete the Stable Degeneration Conjecture (Li15, Li-X.17) in local K-stability theory for klt singularities.

Thank you very much!