Perverse sheaves and positivity of Euler characteristics

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Some motivating questions and results

Conjecture (Singer-Hopf)

Let M be a compact $2d$-dimensional Riemannian manifold. If

1. *M has non-positive sectional, or*
2. *M is aspherical, i.e., the universal covering space of M is contractible,*

then its Euler characteristic satisfies

$$(-1)^d \chi(M) \geq 0.$$

- For any odd-dimensional compact manifold M, $\chi(M) = 0$.
- By Cartan-Hadamard theorem, having non-positive sectional curvature implies aspherical. So the aspherical version is stronger.

Some motivating questions and results

- The conjecture is obvious for Riemann surfaces.

$g(C)$	0	1	≥ 2
curvature	$+$	0	$-$
universal cover	$\mathbb{P}^1$	$\mathbb{C}$	open disc
$-\chi(C)$	-2	0	≥ 2

- If a compact Riemannian 4-manifold M has non-positive sectional curvature, then Gauss-Bonnet theorem implies that $\chi(M) \geq 0$. The aspherical version of the conjecture is open for 4-manifolds.

Some motivating questions and results

Theorem (Gabber-Loeser, Franecki-Kapranov, Schnell, Krämer, . . .)

Let X be a smooth closed subvariety of an affine torus $(\mathbb{C}^*)^n$, or an abelian variety A, (or a semi-abelian variety), then

$$(-1)^{\dim_{\mathbb{C}} X}\chi(X) \geq 0.$$

There are examples of aspherical manifolds as subvarieties of abelian varieties, e.g., all Riemann surfaces of genus at least 1. So this theorem does cover some special cases of the Singer-Hopf conjecture.

Theorem (Seade-Tibăr-Verjovsky, Maxim-Rodriguez-W.)

Let $X \subset \mathbb{C}^n$ be a smooth closed subvariety, and let $H \subset \mathbb{C}^n$ be a general affine hyperplane. Then,

$$(-1)^{\dim_{\mathbb{C}} X}\chi(X \setminus H) \geq 0.$$

Some motivating questions and results

The above two theorems are also related to the complexity of algebraic optimization problems.

Theorem (Franechi-Kapranov, Huh)

Let $X \subset (\mathbb{C}^*)^n$ be a smooth closed subvariety. Then

$$(-1)^{\dim_C X}\chi(X) = \text{the number of critical points of } f_{\mathbf{u}}|_X$$

where $f_{\mathbf{u}} = \prod_{1\leq i\leq n} x_i^{u_i}$ and $u_i \in \mathbb{Z}$ are general.

Theorem (Seade-Tibăr-Verjovsky, Maxim-Rodriguez-W.)

Let $X \subset \mathbb{C}^n$ be a smooth closed subvariety, and let $H \subset \mathbb{C}^n$ be a general affine hyperplane. Then

$$(-1)^{\dim_{\mathbb{C}} X}\chi(X \setminus H) = \text{the number of critical points of } l_{\mathbf{u}}|_X$$

where $l_{\mathbf{u}} = \prod_{1\leq i\leq n} u_i x_i$ is a general linear function.

Some motivating questions and results

Let G be a complex semi-simple linear algebraic group, and let P be a parabolic subgroup. Then G/P is a partial flag variety. The following result was conjectured independently by Knutson–Zinn-Justin and Mihalcea.

Theorem (Schürman-Simpson-W.)

Given any three Schubert cells $X_\lambda^\circ, X_\mu^\circ, X_\nu^\circ$, and two general element $g, h \in G$, let $Z = X_\lambda^\circ \cap gX_\mu^\circ \cap hX_\nu^\circ$. Then

$$(-1)^{\dim Z}\chi(Z) \geq 0.$$

- We know that $\chi(X_\lambda^\circ) = 1$ and $\chi(X_\lambda^\circ \cap gX_\mu^\circ) = 0$ or 1.
- When G/P is the r-step partial flag variety of type A, and when $r \leq 3$, Knutson and Zinn-Justin have a combinatorial formula for $(-1)^{\dim Z}\chi(Z)$, which implies its non-negativity.
- Such $\chi(Z)$ is the structure constant for the multiplication of the Segre-Schwartz-MacPherson classes of these Schubert cells.

Perverse sheaves

A perverse sheaf on a complex algebraic variety is a bounded complex of constructible sheaves subject to certain conditions.

Example

Let X be a smooth complex algebraic variety, and let Y be a smooth closed subvariety of X. Then $\underline{\mathbb{Q}}_Y[\dim Y]$ is a perverse sheaf. Here, $\underline{\mathbb{Q}}_Y[\dim Y]$ is the complex of sheaves

$$\cdots \to 0 \to \underline{\mathbb{Q}}_Y \to 0 \to \cdots$$

where $\underline{\mathbb{Q}}_Y$ is at degree $-\dim Y$. In particular, $\underline{\mathbb{Q}}_X[\dim X]$ is a perverse sheaf on X.

Given a bounded complex of constructible sheaves, $\mathscr{F}^\bullet = (\mathscr{F}^i, d^i)$, the Euler characteristic of $\mathscr{F}^\bullet$ is defined to be

$$\chi(\mathscr{F}^\bullet) := \sum_{i\in\mathbb{Z}} (-1)^i \chi(\mathscr{F}^i).$$

In particular, $\chi(\underline{\mathbb{Q}}_X[\dim X]) = (-1)^{\dim X}\chi(\underline{\mathbb{Q}}_X) = (-1)^{\dim X}\chi(X)$.

Perverse sheaves

The conjectures and results we have seen so far all have generalizations or analogues in terms of perverse sheaves.

Theorem (Gabber-Loeser, Franecki-Kapranov, Schnell, Krämer, ...)

Let $\mathscr{F}^\bullet$ be any perverse sheaf on $(\mathbb{C}^*)^n$. Then $\chi(\mathscr{F}^\bullet) \geq 0$.

Theorem (Seade-Tibăr-Verjovsky)

Let $\mathscr{F}^\bullet$ be any perverse sheaf on $\mathbb{C}^n$, and let H be a general affine hyperplane in $\mathbb{C}^n$. Then, $\chi(\mathscr{F}^\bullet|_{\mathbb{C}^n \setminus H}) \geq 0$.

Theorem (Simpson-Schürman-W.)

Let A and B (e.g., Schubert cells or their generic intersections) are two smooth affine locally closed subvarieties of a partial flag variety G/P. Let $\mathscr{F}^\bullet$ be a perverse sheaf on A, and let $g \in G$ be a general element. Then

$$(-1)^{\mathrm{codim} B}\chi(\mathscr{F}^\bullet|_{A \cap gB}) \geq 0.$$

Perverse sheaves

- The last theorem implies the triple intersection theorem of Schubert cell, by taking $A = X_\lambda^\circ \cap gX_\mu^\circ$, $B = X_\nu^\circ$ and $\mathscr{F}^\bullet = \underline{\mathbb{Q}}_A[\dim A]$.
- In general, the restriction of a perverse sheaf to a (transversal) smooth subvariety becomes a perverse sheaf after shifting by the codimension of the subvariety. More precisely, the in setting of the last theorem,

$$\mathscr{F}^\bullet|_{A\cap gB}[-\mathrm{codim} B]$$

is a perverse sheaf, and the theorem can be reformulated as

$$\chi\left(\mathscr{F}^\bullet|_{A\cap gB}[-\mathrm{codim} B]\right) \geq 0.$$

- If we let $G/P = \mathbb{P}^n$ and $A = B = \mathbb{C}^n$, then for a general $g \in G$, $A \cap gB = \mathbb{C}^n \setminus H$, where H is a general affine hypersurface. So the last theorem implies the earlier theorem stating that for a perverse sheaf $\mathscr{F}^\bullet$ on $\mathbb{C}^n$,

$$\chi(\mathscr{F}^\bullet|_{\mathbb{C}^n\setminus H}) \geq 0.$$

Perverse sheaf analog for Singer-Hopf conjecture

Conjecture (Liu-Maxim-W., Arapura-W.)

Let X be a smooth complex projective variety. If

1. *X is aspherical, or*
2. *X has large fundamental group, i.e., for any Riemann surface C and any non-constant map $f : C \to X$, $f_* : \pi_1(C) \to \pi_1(X)$ has infinite image, (equivalently, the universal cover $\widetilde{X}$ does not contain any positive-dimensional compact analytic subvariety)*

then any perverse sheaf $\mathscr{F}^\bullet$ on X satisfies $\chi(\mathscr{F}^\bullet) \geq 0$.

It is a simple fact that aspherical projective varieties have large fundamental group. However, conversely, only a small portion of varieties with large fundamental group are aspherical. In fact, if X has large fundamental group, then any subvariety of X also has large fundamental group.

Perverse sheaf analog for Singer-Hopf conjecture

Theorem (Liu-Maxim-W., Arapura-W.)

Let X be a smooth complex projective variety. Assume that

1. *the cotangent bundle T^*X is nef (certain positivity condition), or*
2. *the Kähler metric on X has non-positive sectional curvature.*

Then, for any perverse sheaf $\mathscr{F}^\bullet$ on X, $\chi(\mathscr{F}^\bullet) \geq 0$. In particular, the curvature version of Singer-Hopf conjecture holds for all complex smooth projective varieties, which is a result of Gromov, Jost-Zuo, Cao-Xavier.

Standard curvature calculation shows that

$$
\begin{aligned}
& X \text{ has non-positive sectional curvature} \\
\Rightarrow & X \text{ has non-positive holomorphic bisectional curvature} \\
\Rightarrow & T^*X \text{ is nef.}
\end{aligned}
$$

So the nef version of the theorem is stronger.

Theorem (Arapura-W.)

Let X be a smooth complex projective variety with large fundamental group (or simply, aspherical). Suppose that there exists a cohomologically rigid almost faithful semi-simple representation $\rho : \pi_1(X) \to \mathrm{Gl}_n(\mathbb{C})$. Then for any perverse sheaf $\mathscr{F}^\bullet$ on X, $\chi(\mathscr{F}^\bullet) \geq 0$.

Even though the assumption is extremely strong, it can be formulated purely topologically. In contrast, all the previously known results have some geometric assumption.

Ideas of the proofs

Theorem (Liu-Maxim-W.)

Let X be a smooth complex projective variety. If T^*X is nef, then for any perverse sheaf $\mathscr{F}^\bullet$ on X, $\chi(\mathscr{F}^\bullet) \geq 0$.

Kashiwara's index theorem states that $\chi(\mathscr{F}^\bullet)$ can be expressed as certain intersection numbers of algebraic cycles in T^*X. More precisely,

$$\chi(\mathscr{F}^\bullet) = \text{zero section} \cdot CC(\mathscr{F}^\bullet) \text{ in } T^*X,$$

where $CC(\mathscr{F}^\bullet)$ is the characteristic cycle of $\mathscr{F}^\bullet$. For example, $CC(\underline{\mathbb{Q}}_X[\dim X])$ is equal to the zero section of T^*X.

A theorem of Fulton-Lazarsfeld and Demailly-Peternell-Schneider states that in a nef vector bundle E on a smooth complex projective variety X, the intersection number

$$\text{zero section} \cdot \text{any conic cycle} \geq 0.$$

We are done, because all characteristic cycles are conic.

Ideas of the proofs

Theorem (Arapura-W.)

Let X be a smooth complex projective variety with large fundamental group (or simply aspherical). Suppose that there exists a cohomologically rigid almost faithful semi-simple representation $\rho : \pi_1(X) \to \mathrm{Gl}_n(\mathbb{C})$. Then for any perverse sheaf $\mathscr{F}^\bullet$ on X, $\chi(\mathscr{F}^\bullet) \geq 0$.

- Let L_ρ be the local system associated to the representation ρ. By Simpson's theorem, the rigidity of ρ implies that L_ρ supports a variation of complex Hodge structure.
- By a theorem of Esnault-Groechenig, the variation of Hodge structure can be chosen to be defined over $\mathbb{Z}$. Then we have a period map $\varphi : X \to D/\Gamma$.
- The fact that X has large fundamental group implies that φ is finite.
- It is well-known that D/Γ has negative bisectional curvature (along horizontal directions and φ is along horizontal directions). This gives us some desired curvature condition on X, and then we can follow the previous proof of the nef case.

Ideas of the proofs

Theorem (Gabber-Loeser, Franecki-Kapranov)

Let $\mathscr{F}^\bullet$ be any perverse sheaf on $(\mathbb{C}^*)^n$. Then $\chi(\mathscr{F}^\bullet) \geq 0$.

This has several proofs, using Morse theory, intersection theory, or vanishing theorem. We present the vanishing theorem proof here.

Theorem (Gabber-Loeser)

Under the above notations, let L be a general rank one $\mathbb{C}$-local system on $(\mathbb{C}^*)^n$. Then

$$H^i\big((\mathbb{C}^*)^n, \mathscr{F}^\bullet \otimes_{\mathbb{C}} L\big) = 0 \quad \text{for any } i \neq 0.$$

Therefore,

$$\chi(\mathscr{F}^\bullet) = \chi(\mathscr{F}^\bullet \otimes_{\mathbb{C}} L) = \dim H^0\big((\mathbb{C}^*)^n, \mathscr{F}^\bullet \otimes_{\mathbb{C}} L\big) \geq 0.$$

Ideas of the proofs

Theorem (Schürman-Simpson-W.)

Let A and B (e.g., Schubert cells or their generic intersections) are two smooth affine locally closed subvarieties of a partial flag variety G/P. Let $\mathscr{F}^\bullet$ be a perverse sheaf on A, and let $g \in G$ be a general element. Then

$$(-1)^{\operatorname{codim} B}\chi(\mathscr{F}^\bullet|_{A\cap gB}) \geq 0.$$

Let $i_B : A \cap gB \to A$ and $i_A : A \to X$ be the inclusion maps. Then

$$H^i\left(X, i_{A*}i_{B!}(\mathscr{F}^\bullet|_{A\cap gB}[-\operatorname{codim} B])\right) = 0 \quad \text{for any } i \neq 0.$$

Now, we have

$$\begin{aligned}(-1)^{\operatorname{codim} B}\chi(\mathscr{F}^\bullet|_{A\cap gB}) &= \chi\left(i_{A*}i_{B!}(\mathscr{F}^\bullet|_{A\cap gB}[-\operatorname{codim} B])\right) \\ &= \dim H^0\left(X, i_{A*}i_{B!}(\mathscr{F}^\bullet|_{A\cap gB}[-\operatorname{codim} B])\right) \\ &\geq 0.\end{aligned}$$

Question

Can we extend any of the above theorems to varieties over a general field?

Thank you!