Some exact formulas of the KPZ fixed point and directed landscape

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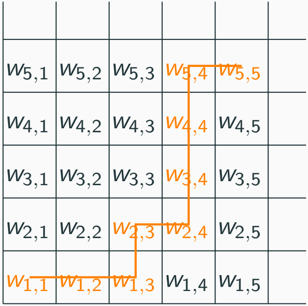
Directed Last Passage Percolation

$w_{5,1}$	$w_{5,2}$	$w_{5,3}$	$w_{5,4}$	$w_{5,5}$	
$w_{4,1}$	$w_{4,2}$	$w_{4,3}$	$w_{4,4}$	$w_{4,5}$	
$w_{3,1}$	$w_{3,2}$	$w_{3,3}$	$w_{3,4}$	$w_{3,5}$	
$w_{2,1}$	$w_{2,2}$	$w_{2,3}$	$w_{2,4}$	$w_{2,5}$	
$w_{1,1}$	$w_{1,2}$	$w_{1,3}$	$w_{1,4}$	$w_{1,5}$	

$w_{i,j} \sim \exp(1)$, i.i.d.

Directed Last Passage Percolation



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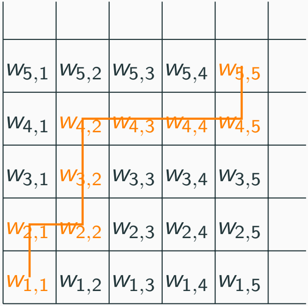

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Directed Last Passage Percolation

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Last passage time

$$L_{\mathbf{p}}(\mathbf{q}) := \max_{\pi:\mathbf{p}\to\mathbf{q}} \sum_{\mathbf{r}\in\pi} w_{\mathbf{r}}$$

where the maximum is taken over all possible up/right lattice path π from $\mathbf{p}$ to $\mathbf{q}$. The path along which the maximum is obtained is called the *geodesic*.

Directed Last Passage Percolation

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More generally,

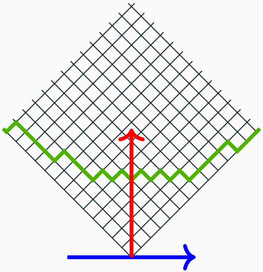
$$L_{\Lambda}(\mathbf{q}) := \max_{\pi:\mathbf{p}\in\Lambda\to\mathbf{q}} \sum_{\mathbf{r}\in\pi} w_{\mathbf{r}},$$

where Λ is an arbitrary down/right lattice path. Λ is called the initial condition.

Equivalent models

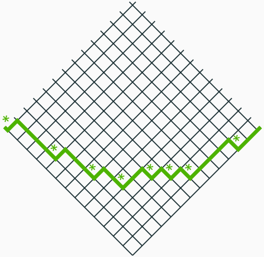
This model is equivalent to the *corner growth model*:

Let $\Omega_t := \{(i,j) \mid L_{(1,1)}(i,j) \leq t\}$.

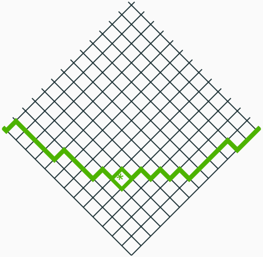


Equivalent models

Evolution of Ω_t:

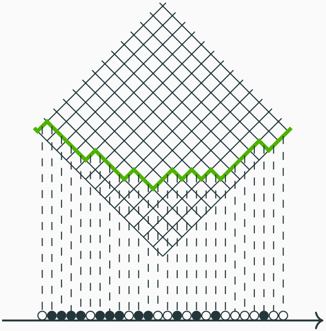


$\longrightarrow$



Equivalent models

It is also equivalent to the totally asymmetric simple exclusion process (TASEP).



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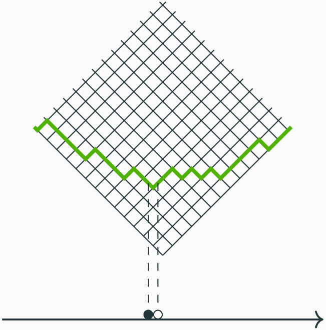
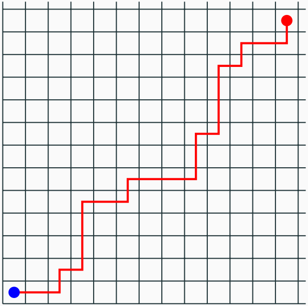


Directed Last Passage Percolation

Theorem (Johanson00)

$$\frac{L_{(1,1)}(\alpha N, N) - c_1 N}{c_2 N^{1/3}} \to \chi_{GUE},$$

in distribution as $N \to \infty$, here α is a positive constant, c_1, c_2 are constants depending on α, and χ_{GUE} is the GUE Tracy-Widom distribution.

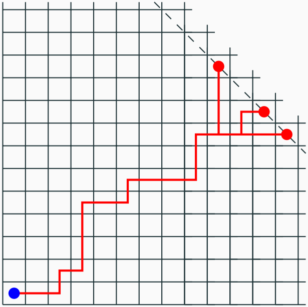


Directed Last Passage Percolation

Theorem (Prahofer-Spohn02, Johansson03)

$$\frac{L_{(1,1)}(N - cxN^{2/3}, N + cxN^{2/3}) - 4N}{c'N^{1/3}} \to \mathcal{A}_2(x) - x^2$$

as $N \to \infty$, where $\mathcal{A}_2(x)$ is called the Airy$_2$ process.

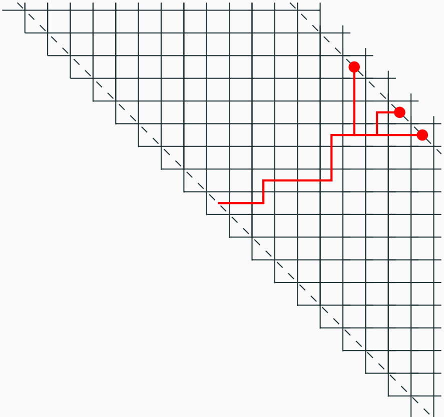


Directed Last Passage Percolation

More generally, for any "nice enough" initial condition Λ, [Matetski-Quastel-Remenik21] proved

$$\frac{L_\Lambda(tN - c_0 x N^{2/3}, tN + c x N^{2/3}) - c_1(tN)}{c_2 N^{1/3}} \to \mathcal{H}(x,t),$$

where $\mathcal{H}(x,t)$ is a space-time random field called *the KPZ fixed point*.



Directed Last Passage Percolation

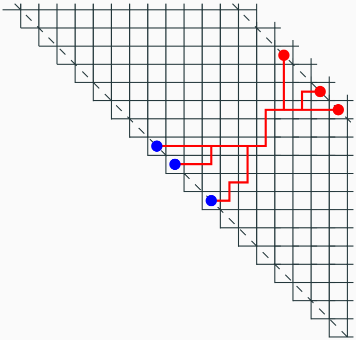
Even more generally, there is a limiting "random metric" $\mathcal{L}(y, s; x, t)$, the *directed landscape* constructed by [Dauvergne-Ortmann-Virag22], such that

$$\frac{L_{(sN - c_0 y N^{2/3}, sN + c_0 y N^{2/3})}(tN - c_0 x N^{2/3}, tN + c_0 x N^{2/3}) - c_1(t-s)N}{c_2 N^{1/3}}$$

$$\to \mathcal{L}(y, s; x, t).$$

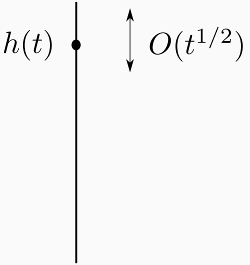
This convergence was proved by [Dauvergne-Virag21].

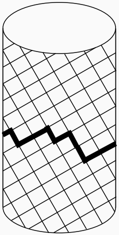
Directed Last Passage Percolation

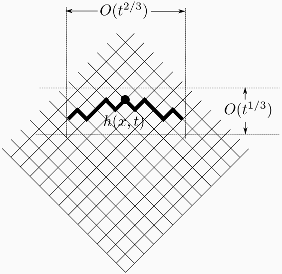


An illustration of the convergence of the directed last passage percolation to the directed landscape. Here we fix two times $s = 0$ and $t = 1$. The last passage times from the blue boxes (indexed by the parameter y) to the red boxes (indexed by the parameter x) converge to $\mathcal{L}(y, 0; x, 1)$, which is also called the Airy sheet $\mathcal{S}(y, x)$.

A crossover between Gaussian and KPZ universalities







What was known before for this crossover model

Let L denote the perimeter of the base circle, or equivalently the period of periodic TASEP model.

Three cases:

1. When L is sufficiently large compared to t, it behaves like the (1+1)d corner growth model (sub-relaxation time scale).
2. When t is sufficiently large compared to L, it behaves like the 1d growth model (super-relaxation time scale)?
3. The crossover occurs when $t = O(L^{3/2})$ (relaxation time scale) (Gwa and Spohn 1992)

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The conjectured periodic KPZ fixed point and periodic directed landscape are the limits ot this model in the relaxation time scale.

Some basic properties of the KPZ fixed point and directed landscape

Some basic properties of the KPZ fixed point and directed landscape

The KPZ fixed point and directed landscape are conjectured to be the universal limiting fields for all the models in a broad class of models, the Kardar-Parisi-Zhang (KPZ) universality class. Hence it is an important problem to well understand these random fields.

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The KPZ Universality Class: random growth models, interacting particle systems, random polymers, random matrices, etc.

Some basic properties of the KPZ fixed point and directed landscape

Relation between the KPZ fixed point and directed landscape, proved by [Nica-Quastel-Remenik20]

$$\mathcal{H}(x,t) = \sup_y \{g(y) + \mathcal{L}(y,0;x,t)\} \tag{2.1}$$

where $g(y)$ is the initial condition $\mathcal{H}(y,0)$.

Some basic properties of the KPZ fixed point and directed landscape

Composition law:

$$\mathcal{L}(z, r; x, t) = \sup_{y \in \mathbb{R}} \{\mathcal{L}(z, r; y, s) + \mathcal{L}(y, s; x, t)\}$$

Some basic properties of the KPZ fixed point and directed landscape

Composition law:

$$\mathcal{L}(z,r;x,t) = \sup_{y\in\mathbb{R}}\{\mathcal{L}(z,r;y,s)+\mathcal{L}(y,s;x,t)\}$$

Reverse triangle inequality:

$$\mathcal{L}(z,r;x,t) \geq \mathcal{L}(z,r;y,s)+\mathcal{L}(y,s;x,t)$$

Some basic properties of the KPZ fixed point and directed landscape

The *geodesic* $\pi_{y,s;x,t}$ from a point (y, s) to another point (x, t) if $s < t$ is a continuous path $\pi = \pi_{y,s;x,t}$ which maximizes the following length with respect to the directed landscape $\mathcal{L}$

$$\inf_{k\in\mathbb{N}} \inf_{s=s_0<s_1<\cdots<s_k=t} \sum_{i=1}^{k} \mathcal{L}(\pi(s_{i-1}), s_{i-1}; \pi(s_i), s_i).$$

Such a geodesic exists and is unique almost surely [Dauvergne-Ortmann-Virag22].

Some basic properties of the KPZ fixed point and directed landscape

Symmetry properties

(1) $\mathcal{L}(y,s;x,t) \overset{d}{=} \mathcal{L}(y,s+r;x,t+r)$.

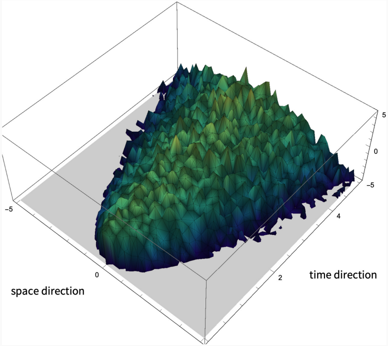
(2) $\mathcal{L}(y,s;x,t) \overset{d}{=} \mathcal{L}(y+c,s;x+c,t)$.

(3) $\mathcal{L}(y,s;x,t) \overset{d}{=} \mathcal{L}(-x,-t;-y,-s)$.

(4) $\mathcal{L}(y,s;x,t) + (x-y)^2/(t-s) \overset{d}{=}$
$\mathcal{L}(y+cs,s;x+ct,t) + (x+ct-y-cs)^2/(t-s)$.

(5) $\mathcal{L}(y,s;x,t) \overset{d}{=} \epsilon\mathcal{L}(\epsilon^{-2}y,\epsilon^{-3}s;\epsilon^{-2}x,\epsilon^{-3}t)$.

Some basic properties of the KPZ fixed point and directed landscape



An illustration of the KPZ fixed point $\mathcal{H}(x, t)$ with the step initial condition, which is equivalent to $\mathcal{L}(0, 0; x, t)$. The gray area is due to the cutoff at the height $= -5$.

Exact formulas

Some known exact formulas

We will mainly focus on the exact formulas for the distributions of the KPZ fixed point and directed landscape.

For the KPZ fixed point, we call $g(y) = \mathcal{H}(y, 0)$ the *initial condition*. There are several classic initial conditions:

(1) Step (or narrow-wedge) initial condition: $g(0) = 0$, and $g(y) = -\infty$ for $y \neq 0$.

(2) Flat initial condition: $g(y) = 0$ for all y.

(3) Brownian initial condition: $g(y)$ is a two-sided Brownian motion.

(4) Step-flat (or wedge-flat) initial condition: $g(y) = 0$ for all $y \geq 0$ (or $y \leq 0$), and $g(y) = -\infty$ elsewhere.

(5) Step-Brownian (or wedge-Brownian) initial condition: $g(y)$ is a Brownian motion for $y \geq 0$ (or $y \leq 0$), and $g(y) = -\infty$ elsewhere.

(6) Flat-Brownian initial condition: $g(y)$ is a Brownian motion for $y \geq 0$ (or $y \leq 0$), and $g(y) = 0$ elsewhere.

Some known exact formulas

One point distribution of $\mathcal{H}_g(x,t)$ (for fixed x and t):

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One point distribution of $\mathcal{H}_g(x,t)$ (for fixed x and t):

They are the Tracy-Widom distributions and their analogs:

(1) Step IC: GUE Tracy-Widom distribution [Baik-Deift-Johansson99]

(2) Flat IC: GOE Tracy-Widom distribution [Baik-Rains01,Ferrari-Spohn05]

(3) Brownian IC: Baik-Rains distribution [Baik-Rains00]

(4) Other classic IC: [Baik-Rains00], [Borodin-Ferrari-Sasamoto09].

(5) General IC: [Corwin-Liu-Wang16], [Matetski-Quastel-Remenik21].

Some known exact formulas

GUE Tracy-Widom distribution $\mathbb{P}(\chi_{GUE} \leq h)$

$$F_{GUE}(h) = \det(I - A_h)$$

where A_h is the *Airy kernel* on $L^2(h, \infty)$ defined by

$$A_h(x,y) = \begin{cases} \dfrac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(y)\mathrm{Ai}'(x)}{x-y}, & \text{if} x \neq y, \\ \mathrm{Ai}'(x)^2 - x(\mathrm{Ai}(x))^2, & \text{if} x = y \end{cases}$$

and $\mathrm{Ai}(x) = \frac{1}{2\pi\mathrm{i}} \int_{\infty e^{-2\pi\mathrm{i}/3}}^{\infty e^{2\pi\mathrm{i}}/3} e^{-\frac{z^3}{3}+zx} \mathrm{d}z$ is the Airy function.

Some known exact formulas

The spatial processes $\mathcal{H}_g(\cdot, t)$ (for fixed t):

They are the Airy processes and their analogs:

(1) Step IC: [Prahofer-Spohn02] ,[Johansson03]

(2) Flat IC: [Borodin-Ferrari-Prahofer-Sasamoto07]

(3) Brownian IC: [Baik-Ferrari-Peche10]

(4) Other classic IC: [Borodin-Ferrari-Sasamoto08], [Borodin-Ferrari-Sasamoto08], [Borodin-Ferrari-Sasamoto09]

(5) General IC: [Matetski-Quastel-Remenik21]

Some known exact formulas

The limiting process for Step IC is $\mathcal{A}_2(x) - x^2$, where $\mathcal{A}_2(x)$ is defined by its finite-dimensional distributions

$$\mathbb{P}\left(\cap_{\ell=1}^{m}\{\mathcal{A}_2(x_\ell) \leq h_\ell\}\right) = \det\left(I - \chi_h K_{Ai} \chi_h\right)\big|_{L^2(\{x_1,\cdots,x_m\}\times\mathbb{R})}$$

where $\chi_h(x_k, \zeta) = 1_{\zeta > h_k}$ and K_{Ai} is the *extended Airy Kernel* defined by

$$K_{Ai}(x,\xi;y,\eta) = \begin{cases} \displaystyle\int_0^\infty \mathrm{Ai}(\xi+\lambda)\mathrm{Ai}(\eta+\lambda)e^{-\lambda(x-y)}\mathrm{d}\lambda, & \text{if} x \geq y, \\ \displaystyle -\int_{-\infty}^0 \mathrm{Ai}(\xi+\lambda)\mathrm{Ai}(\eta+\lambda)e^{-\lambda(x-y)}, & \text{if} x < y. \end{cases}$$

Some known exact formulas

The space-time random field $\mathcal{H}_g(\cdot, \cdot)$:

(1) Step IC: [Johansson-Rahman21], [Liu22]

(2) Flat IC: [Liu22]

Some known exact formulas

The multipoint distribution of the KPZ fixed point $\mathcal{H}(x,t)$ is given by

$$\begin{aligned}&\mathbb{P}\left(\bigcap_{\ell=1}^{m}\{\mathcal{H}(x_\ell,t_\ell)\le h_\ell\}\right)\\&=\oint\cdots\oint\left[\prod_{\ell=1}^{m-1}\frac{1}{1-z_\ell}\right]\mathrm{D}(z_1,\cdots,z_{m-1})\frac{\mathrm{d}z_1}{2\pi\mathrm{i}z_1}\cdots\frac{\mathrm{d}z_{m-1}}{2\pi\mathrm{i}z_{m-1}}\end{aligned}$$

Here, the integral contours are circles centered at the origin and of radii less than 1. The function D will be given later.

Some known exact formulas

$$\mathrm{D}_{\mathrm{step}}(z_1,\cdots,z_{m-1}) = \sum_{\boldsymbol{n}\in(\mathbb{Z}_{\geq 0})^m} \frac{1}{(\boldsymbol{n}!)^2}\mathrm{D}_{\boldsymbol{n},\mathrm{step}}(z_1,\cdots,z_{m-1})$$

with

$$\begin{aligned}
&\mathrm{D}_{\boldsymbol{n},\mathrm{step}}(z_1,\cdots,z_{m-1}) = (-1)^{n_1+\cdots+n_m} \\
&\quad \prod_{\ell=2}^{m}\prod_{i_\ell=1}^{n_\ell}\left[\frac{1}{1-z_{\ell-1}}\int_{\mathrm{C}_{\ell,\mathrm{L}}^{\mathrm{in}}}\frac{\mathrm{d}\xi_{i_\ell}^{(\ell)}}{2\pi\mathrm{i}} - \frac{z_{\ell-1}}{1-z_{\ell-1}}\int_{\mathrm{C}_{\ell,\mathrm{L}}^{\mathrm{out}}}\frac{\mathrm{d}\xi_{i_\ell}^{(\ell)}}{2\pi\mathrm{i}}\right]\cdot\prod_{i_1=1}^{n_1}\int_{\mathrm{C}_{1,\mathrm{L}}}\frac{\mathrm{d}\xi_{i_1}^{(1)}}{2\pi\mathrm{i}} \\
&\quad \prod_{\ell=2}^{m}\prod_{i_\ell=1}^{n_\ell}\left[\frac{1}{1-z_{\ell-1}}\int_{\mathrm{C}_{\ell,\mathrm{R}}^{\mathrm{in}}}\frac{\mathrm{d}\eta_{i_\ell}^{(\ell)}}{2\pi\mathrm{i}} - \frac{z_{\ell-1}}{1-z_{\ell-1}}\int_{\mathrm{C}_{\ell,\mathrm{R}}^{\mathrm{out}}}\frac{\mathrm{d}\eta_{i_\ell}^{(\ell)}}{2\pi\mathrm{i}}\right]\cdot\prod_{i_1=1}^{n_1}\int_{\mathrm{C}_{1,\mathrm{R}}}\frac{\mathrm{d}\eta_{i_1}^{(1)}}{2\pi\mathrm{i}} \\
&\qquad \left[\prod_{\ell=1}^{m}\frac{(\Delta(\boldsymbol{\xi}^{(\ell)}))^2(\Delta(\boldsymbol{\eta}^{(\ell)}))^2}{(\Delta(\boldsymbol{\xi}^{(\ell)};\boldsymbol{\eta}^{(\ell)}))^2}\mathrm{f}_\ell(\boldsymbol{\xi}^{(\ell)})\mathrm{f}_\ell(\boldsymbol{\eta}^{(\ell)})\right] \\
&\qquad \cdot\left[\prod_{\ell=1}^{m-1}\frac{\Delta(\boldsymbol{\xi}^{(\ell)};\boldsymbol{\eta}^{(\ell+1)})\Delta(\boldsymbol{\eta}^{(\ell)};\boldsymbol{\xi}^{(\ell+1)})}{\Delta(\boldsymbol{\xi}^{(\ell)};\boldsymbol{\xi}^{(\ell+1)})\Delta(\boldsymbol{\eta}^{(\ell)};\boldsymbol{\eta}^{(\ell+1)})}(1-z_\ell)^{n_\ell}\left(1-\frac{1}{z_\ell}\right)^{n_{\ell+1}}\right].
\end{aligned}$$

Some known exact formulas

Here

$$\boldsymbol{n} = (n_1, \cdots, n_m), \quad \boldsymbol{n}! = n_1! \cdots n_m!,$$

$$\boldsymbol{\xi}^{(\ell)} = (\xi_1^{(\ell)}, \cdots, \xi_{n_\ell}^{(\ell)}), \quad \boldsymbol{\eta}^{(\ell)} = (\eta_1^{(\ell)}, \cdots, \eta_{n_\ell}^{(\ell)}),$$

$$f_\ell(\boldsymbol{\xi}^{(\ell)}) = \prod_{i_\ell=1}^{n_\ell} e^{-\frac{1}{3}(t_\ell - t_{\ell-1})(\xi_{i_\ell}^{(\ell)})^3 + (x_\ell - x_{\ell-1})(\xi_{i_\ell}^{(\ell)})^2 + (h_\ell - h_{\ell-1})\xi_{i_\ell}^{(\ell)}},$$

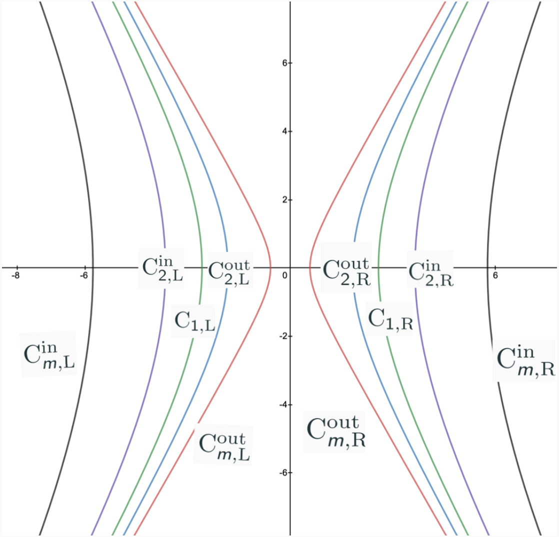
$$f_\ell(\boldsymbol{\eta}^{(\ell)}) = \prod_{i_\ell=1}^{n_\ell} e^{\frac{1}{3}(t_\ell - t_{\ell-1})(\eta_{i_\ell}^{(\ell)})^3 - (x_\ell - x_{\ell-1})(\eta_{i_\ell}^{(\ell)})^2 - (h_\ell - h_{\ell-1})\eta_{i_\ell}^{(\ell)}},$$

with $\tau_0 = x_0 = h_0 = 0$, and

$$\Delta(W) := \prod_{i<j}(w_j - w_i), \quad \Delta(W; W') := \prod_{i,i'}(w_i - w'_{i'})$$

for all vectors $W = (w_1, \cdots, w_n)$, $W' = (w'_1, \cdots, w'_{n'})$.

Some known exact formulas



Some known exact formulas

The one-point distribution of the geodesic

(1) Point-to-point geodesic: [Liu22]

(2) Line-to-point geodesic: [Liu23+]

One-point distribution of the geodesic

Let $\Pi(s) = \Pi_{0,0;0,1}(s)$, $0 \leq s \leq 1$, be the geodesic from $(0,0)$ to $(0,1)$ in the directed landscape. Then the joint density of

$$\mathcal{L}(0,0;s,\Pi(s)), \mathcal{L}(s,\Pi(s);0,1), \Pi(s)$$

is given by

$$2\mathrm{p}\left(\ell_1 + \frac{x^2}{s}, \ell_2 + \frac{x^2}{1-s}, 2x; s\right)$$

where p is defined below.

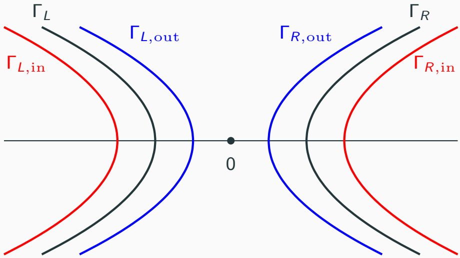
The function $\mathrm{p}(\mathrm{s}_1, \mathrm{s}_2, \mathrm{x}; \gamma)$ is given by

$$\begin{aligned}&\mathrm{p}(\mathrm{s}_1, \mathrm{s}_2, \mathrm{x}; \gamma)\\&= \oint_0 \frac{\mathrm{d}\mathrm{z}}{2\pi \mathrm{i}(1-\mathrm{z})^2} \sum_{k_1, k_2 \geq 1} \frac{1}{(k_1! k_2!)^2} \mathrm{T}_{k_1, k_2}(\mathrm{z}; \mathrm{s}_1, \mathrm{s}_2, \mathrm{x}; \gamma)\end{aligned}$$

where

$$\begin{aligned}&\mathrm{T}_{k_1, k_2}(\mathrm{z}; \mathrm{s}_1, \mathrm{s}_2, \mathrm{x}; \gamma)\\&= \prod_{i_1=1}^{k_1} \left(\frac{1}{1-\mathrm{z}} \int_{\Gamma_{L,\mathrm{in}}} \frac{\mathrm{d}\xi_{i_1}^{(1)}}{2\pi\mathrm{i}} - \frac{\mathrm{z}}{1-\mathrm{z}} \int_{\Gamma_{L,\mathrm{out}}} \frac{\mathrm{d}\xi_{i_1}^{(1)}}{2\pi\mathrm{i}} \right)\\&\quad \prod_{i_1=1}^{k_1} \left(\frac{1}{1-\mathrm{z}} \int_{\Gamma_{R,\mathrm{in}}} \frac{\mathrm{d}\eta_{i_1}^{(1)}}{2\pi\mathrm{i}} - \frac{\mathrm{z}}{1-\mathrm{z}} \int_{\Gamma_{R,\mathrm{out}}} \frac{\mathrm{d}\eta_{i_1}^{(1)}}{2\pi\mathrm{i}} \right)\\&\quad \prod_{i_2=1}^{k_2} \int_{\Gamma_L} \frac{\mathrm{d}\xi_{i_2}^{(2)}}{2\pi\mathrm{i}} \prod_{i_2=1}^{k_2} \int_{\Gamma_R} \frac{\mathrm{d}\eta_{i_2}^{(2)}}{2\pi\mathrm{i}} \quad \boxed{\text{the integrand}}\end{aligned}$$

The function $\mathrm{p}(\mathrm{s}_1, \mathrm{s}_2, \mathrm{x}; \gamma)$



The contours

The function $\mathrm{p}(\mathrm{s}_1, \mathrm{s}_2, \mathrm{x}; \gamma)$

$$\boxed{\text{the integrand}} = (1-\mathrm{z})^{k_2}(1-\mathrm{z}^{-1})^{k_1} \cdot \prod_{i_1=1}^{k_1} \frac{\mathrm{f}_1(\xi_{i_1}^{(1)})}{\mathrm{f}_1(\eta_{i_1}^{(1)})} \prod_{i_2=1}^{k_2} \frac{\mathrm{f}_2(\xi_{i_2}^{(2)})}{\mathrm{f}_2(\eta_{i_2}^{(2)})}$$

$$\prod_{\ell=1}^{2} \frac{\Delta(\boldsymbol{\xi}^{(\ell)})^2 \Delta(\boldsymbol{\eta}^{(\ell)})^2}{\Delta(\boldsymbol{\xi}^{(\ell)}; \boldsymbol{\eta}^{(\ell)})^2} \cdot \frac{\Delta(\boldsymbol{\xi}^{(1)}; \boldsymbol{\eta}^{(2)})\Delta(\boldsymbol{\eta}^{(1)}; \boldsymbol{\xi}^{(2)})}{\Delta(\boldsymbol{\xi}^{(1)}; \boldsymbol{\xi}^{(2)})\Delta(\boldsymbol{\eta}^{(1)}; \boldsymbol{\eta}^{(2)})}$$

$$\mathrm{H}(\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}; \boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}),$$

with

$$\mathrm{f}_1(\zeta) = e^{-\frac{\gamma}{3}\zeta^3 - \frac{1}{2}\mathrm{x}\zeta^2 + (\mathrm{s}_1 - \frac{\mathrm{x}^2}{4\gamma})\zeta}, \mathrm{f}_2(\zeta) = e^{-\frac{1-\gamma}{3}\zeta^3 + \frac{1}{2}\mathrm{x}\zeta^2 + (\mathrm{s}_2 - \frac{\mathrm{x}^2}{4(1-\gamma)})\zeta},$$

and

$$\mathrm{H}(\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}; \boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}) = \frac{1}{12}\mathrm{S}_1^4 + \frac{1}{4}\mathrm{S}_2^2 - \frac{1}{3}\mathrm{S}_1\mathrm{S}_3,$$

where

$$\mathrm{S}_\ell = \sum \left((\xi_{i_1}^{(1)})^\ell - (\eta_{i_1}^{(1)})^\ell \right) - \sum \left((\xi_{i_2}^{(2)})^\ell - (\eta_{i_2}^{(2)})^\ell \right).$$

Exact formulas of the conjectured periodic KPZ fixed point

One-point distribution

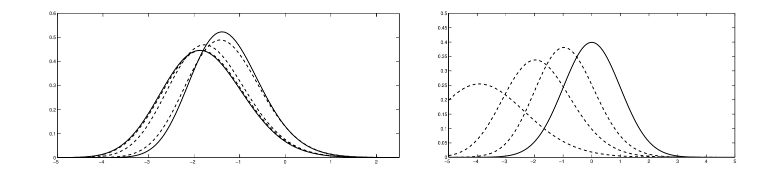
- Step IC: [Baik-Liu18]
- Flat IC: [Baik-Liu18]
- Stationary IC: [Liu18]
- General IC: [Baik-Liu21]

Multi-point distribution

- Step IC: [Baik-Liu19]
- General IC: [Baik-Liu21]

The formulas are given in terms of Fredholm determinant with kernel on the space related to the roots of $e^{-\xi^2/2} = z$.

Exact formulas of the conjectured periodic KPZ fixed point



Left: The dashed curves are density functions of $F_{step}(\tau^{1/3}x - \frac{\gamma^2}{4\tau}; \gamma, \tau)$ for fixed $\tau = 0.1$ and three different values of $\gamma = 0.2, 0.4, 0.5$ from left to right. The solid curves are the density functions of $F_{GUE}(x)$ (left) and $(F_{GUE}(x))^2$ (right).

Right: The dashed curves are the density functions of $F_{step}\left(-\tau + \frac{\pi^{1/4}}{\sqrt{2}}\tau^{1/2}x; \gamma, \tau\right)$ for fixed $\gamma = 0.2$ and three values of $\tau = 0.05, 0.25, 1$ from left to right. The solid curve is the standard Gaussian density function.

Applications

Rigidity of the geodesic

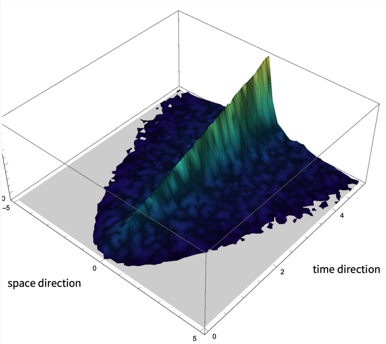
Denote $\mathcal{L}(y, s; x, t)$ the directed landscape, the limiting four-parameter random field of the directed last passage percolation. We fix two points $(0, 0)$ and $(0, 1)$. Denote $\Pi(s)$ the geodesic from $(0, 0)$ to $(0, 1)$. We also denote $\mathcal{L}(s) = \mathcal{L}(0, 0; \Pi(s), s)$

Theorem (Liu22)
The random variables

$$\frac{2\Pi(s)L^{1/4}}{\sqrt{s(1-s)}}, \frac{\mathcal{L}(s) - sL}{\sqrt{s(1-s)}L^{1/4}}$$

conditioned on $\mathcal{L}(1) = L \to \infty$, converge to two independent standard Gaussian random variables in distribution.

Rigidity of the geodesic



An illustration of the rigidity of the geodesic conditioned on a large height ($= L$) of the directed landscape at one location $\mathcal{L}(0,0;0,5)$. The field is of the hill-shape with a straight ridge (with width of $O(L^{-1/4})$) from the point $(0,0)$ to $(0,5)$, and the geodesic is on the ridge with Gaussian fluctuation. The height along the geodesic $\mathcal{L}(0,0;\pi(t),t)$ fluctuates of order $O(L^{1/4})$ and has the Gaussian fluctuation as well.

Rigidity of the gedesic

We remark that this result was unexpected before. However, the rigidity of the geodesic was observed before by [Basu-Ganguly19]. Loosely speaking (their results are indeed for the directed last passage percolation but we take the KPZ limit of their results which is expected to be valid) if we heuristically convert the results of [Basu-Ganguly19] into our language, their results indicate that $\Pi(s)$ has an fluctuation order $O(L^{-1/4+o(1)})$.

A conditional limiting field

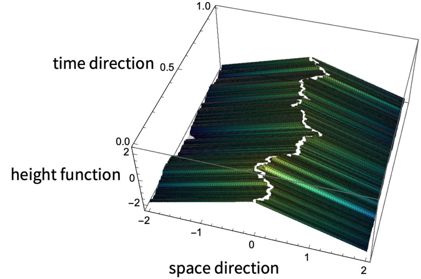
In [Liu-Wang22], we proved that for all $T > 0$, as $L \to \infty$,

$$\mathrm{Law}\left(\left\{ \frac{\mathcal{H}(\frac{xT^{3/4}}{\sqrt{2}L^{1/4}}, tT) - t\mathcal{H}(0,T)}{\sqrt{2}T^{1/4}L^{1/4}} \right\}_{x\in\mathbb{R}, t\in(0,1)} \middle| \, \mathcal{H}(0,T) = L \right)$$

$$\to \begin{cases} \mathrm{Law}\left(\left\{ \min\left\{ \mathbb{B}_1^{\mathrm{br}}(t) + x, \mathbb{B}_2^{\mathrm{br}}(x) - x \right\} \right\}_{x\in\mathbb{R}, t\in(0,1)} \right), \\ \mathrm{Law}\left(\left\{ \min\left\{ \mathbb{B}_1^{\mathrm{br}}(t) + x + \frac{1-t}{\sqrt{2}} Z, \mathbb{B}_2^{\mathrm{br}}(x) - x - \frac{1-t}{\sqrt{2}} Z \right\} \right\}_{x\in\mathbb{R}, t\in(0,1)} \right), \end{cases}$$

where the convergence is in the sense of convergence of finite-dimensional distributions, and on the right-hand side, $\{\mathbb{B}_1^{\mathrm{br}}(t)\}_{t\in[0,1]}$ and $\{\mathbb{B}_2^{\mathrm{br}}(t)\}_{t\in[0,1]}$ denote two i.i.d. copies of Brownian bridge over interval $[0,1]$, and in the case with flat initial condition Z is a standard normal random variable independent of $\mathbb{B}_1^{\mathrm{br}}, \mathbb{B}_2^{\mathrm{br}}$.

A conditional limiting field



An illustration of the rescaled random field on the ridge of the KPZ fixed point (for the step initial condition): It behaves like a Brownian bridge along the time direction, and straight lines along the space direction. The highest curve (along the time direction) is a Brownian bridge.

A conditional limiting field

The best related related results are (if we "translate" into the language of the KPZ fixed point) by [Lamarre-Lin-Tsai21], [Ganguly-Hegde22]

$$\mathcal{H}(x,1) \approx L - 2|x|L^{1/2}$$

when $|x| < L^{1/2}$ conditioning on $\mathcal{H}(x,1) = L \to \infty$.

A conditional limiting field

What happens for $\mathcal{H}(x, T+t)$ if we condition on $\mathcal{H}(0, T) = L \to \infty$? It turns out that for the step initial condition ([Nissim-Zhang22] proved the one-point distribution case)

$$\mathrm{Law}\left(\mathcal{H}(x, T+t) - \mathcal{H}(0, T) \mid \mathcal{H}(0, T) = L\right) \to \hat{\mathcal{H}}(x, t)$$

where $\hat{\mathcal{H}}$ is the KPZ fixed point with step initial condition independent of $\mathcal{H}$.

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where $\hat{\mathcal{H}}$ is the KPZ fixed point with step initial condition independent of $\mathcal{H}$.

Moreover, there is a critical region where the limit of

$$\mathrm{Law}\left(L^{1/2}\left(\mathcal{H}(xL^{-1}, T + tL^{-3/2}) - \mathcal{H}(0, T = 1)\right) \,\middle|\, \mathcal{H}(0, T = 1) = L\right)$$

converges and it is the crossover between the KPZ fixed point and the Brownian-like field. This is a joint work with Zhang ([Liu-Zhang23+]).

A conditional limiting field in periodic KPZ fixed point

Let $\mathcal{H}_p^{per}(x,t)$ be the $p/2$-periodic KPZ fixed point. It satisfies the periodicity condition $\mathcal{H}_p^{per}(x,t) = \mathcal{H}_p^{per}(x+p/2,t)$, and the scaling invariance $p^{-1/2}\mathcal{H}_p^{per}(px, p^{3/2}t) \stackrel{d}{=} q^{-1/2}\mathcal{H}_q^{per}(qx, q^{3/2}t)$.

In [Baik-Liu23+], we proved that

$$\mathrm{Law}\left(\left\{\frac{\mathcal{H}_p^{per}(\frac{x}{\sqrt{2}L^{1/4}},t) - t\mathcal{H}(0,T)}{\sqrt{2}L^{1/4}}\right\}_{x\in\mathbb{R},t\in(0,1)} \,\middle|\, \mathcal{H}(0,1)=L\right)$$
$$\to \mathrm{Law}\left(\left\{s_r\left(\mathbb{B}_1^{\mathrm{br}}(t)+x, \mathbb{B}_2^{\mathrm{br}}(x)-x\right)\right\}_{x\in\mathbb{R},t\in(0,1)}\right),$$

where $r = pL^{1/4}/\sqrt{2}$.

A conditional limiting field in periodic KPZ fixed point

We define an equivalence relation $\sim_r$ on $\mathbb{R}^2$ by the condition that $(x, y) \sim_r (x', y')$ if and only if $(x, y) = (x', y') + k(r, -r)$ for some integer k.

$(\mathbb{B}_1^{\mathrm{br}}(t), \mathbb{B}_2^{\mathrm{br}}(t))$ is a two-dimensional Brownian bridge in the space $\mathcal{R}_r = \mathbb{R}^2 / \sim_r$ satisfying $(\mathbb{B}_1^{\mathrm{br}}(0), \mathbb{B}_2^{\mathrm{br}}(0)) = (\mathbb{B}_1^{\mathrm{br}}(1), \mathbb{B}_2^{\mathrm{br}}(1)) = (0, 0) \in \mathcal{R}_r$.

The height function $s_r : \mathcal{R}_r \to \mathbb{R}$ is defined by

$$s_r(x, y) = \max\left\{ s \in \mathbb{R} : \left[\frac{x - s}{r}\right] + \left[\frac{y - s}{r}\right] \geq 0 \right\}$$

where $[a]$ denotes the largest integer less than or equal to a.

Future questions

Future questions

There are many questions along this line of research.

(1) Find the exact formulas of the multi-point distribution of $\mathcal{H}_g(x,t)$ when g is (1) Brownian, (2) step-flat, (3) step-Brownian, (4) flat-Brownian, and (5) a general function.

(2) Investigate the limiting behavior $\mathcal{H}(x,t)$ when $\mathcal{H}(0,T) = -L$ goes to negative infinity.

(3) Find the exact formulas of the multi-point distribution of the geodesic $\pi(s)$ in the directed landscape.

(4) Analogs in the periodic KPZ fixed point and periodic directed landscape.

Thank you!