IV. Cotangent complexes

1. The cotangent complex formalism

Goal: derive Kähler differentials

Recall: A commutative ring, M A-module

A derivation from A to M is a map \( d : A \to M \) satisfying

\[
d(x + y) = dx + dy \quad d(xy) = xdy + ydx
\]

Let \( \text{Der}(A, M) \) be the abelian group of derivations from A to M.

Fix A, the functor \( M \mapsto \text{Der}(A, M) \) is co-represented by an A-module \( \Omega_A \), called the A-module of absolute Kähler differentials.

Explicitly, \( \Omega_A = \text{free module generated by the symbols } \{ dx \}_{x \in A} \)

/relations \( d(x + y) = dx + dy, \ d(xy) = xdy + ydx, \quad x, y \in A \).

Reformulation:

Let \( B = A \oplus M \), equipped with the ring structure given by

\[
(a, m) (a', m') = (aa', am' + a'm)
\]
called a trivial square-zero extension.

Then \( \text{Der}(A, M) = \text{sections of the projection map } A \oplus M \to A \).

Let \( \text{Ring} = \{ \text{commutative rings} \} \)

\( \text{Ring}^+ = \{ (A, M) | A \text{ is a commutative ring, } M \text{ is an A-module} \} \)
Mor: \((f, f'): (A, M) \to (B, N)\)
\[ \mapsto \text{A ring morphism } \quad f': M \to N \text{ map of } A\text{-modules.} \]

\[ G: \text{Ring}^+ \to \text{Ring} \quad (A, M) \mapsto A \oplus M \text{ trivial square-zero extension.} \]

\(G\) admits a left adjoint \(F: A \mapsto (A, \Omega_A)\).

### Steps for generalizing the above construction to derived geometry:

1. Generalize trivial square-zero extension
2. Generalize \(\text{Ring}^+\) to \(\mathcal{C}^+\) for any presentable \(\infty\)-category \(\mathcal{C}\).
   \(L: T\mathcal{C}\) called the tangent bundle to \(\mathcal{C}\).
3. Define the cotangent complex functor \(L: \mathcal{C} \to T\mathcal{C}\) via adjunction.
4. Define derivation via the tangent correspondence to \(\mathcal{C}\).

#### 1.1 Trivial square-zero extension

**Goal:** Given \(A\ \text{E}_\infty\)-ring, \(M \in \text{Mod}_A\), construct the trivial square-zero extension \(A \oplus M\). We want a functorial construction, i.e. a trivial square-zero extension functor
\[ G: \text{Mod}_A \to \text{CAlg}/A. \]

**Construction:** Let \(X\) be an object of \(\text{Sp}(\text{CAlg}/A)\). Then the \(0^{th}\)-space \(\Omega^0(X)\) is a pointed object of \(\text{CAlg}/A\), i.e. an \(\text{E}_\infty\)-ring \(B\) which fits into a commutative diagram.
Note that the fiber of $f$ inherits the structure of an $A$-module functor $F': \text{Sp}(\text{CAlg/}A) \to \text{Mod}_A$

Define the trivial square-zero extension functor $G$ to be the composition $\text{Mod}_A \xrightarrow{\sim} \text{Sp}(\text{CAlg/}A) \xrightarrow{\Omega^\infty} \text{CAlg/}A$

Denote $A\oplus M := G(M)$.

Forgetting the algebra structure, $A\oplus M$ is canonically identified with the coproduct of $A$ and $M$.

The multiplication on $\pi_*(A\oplus M)$ is given on homogeneous elements by the formula $(a, m)(a', m') = (aa', am' + (-1)^{|a||m|} a'm)$. In particular, if $A$ and $M$ are discrete, then $A\oplus M$ is identified with the classical trivial square-zero extension.

1.2 Stable envelopes and tangent bundles

Idea: make the above construction in families, i.e. fibrewise stabilization.

Let's first give a characterization of the stabilization $\text{Sp}(C)$.
Def: $\mathcal{C}$ presentable $\infty$-category. A stable envelope of $\mathcal{C}$ is a functor $u: \mathcal{C}' \rightarrow \mathcal{C}$ st.

(i) $\mathcal{C}'$ is a presentable stable $\infty$-category.

(ii) $u$ admits a left adjoint

(iii) $\mathcal{E}$ presentable stable $\infty$-category, composition with $u$ induces an equivalence of $\infty$-cats.

$$RFun(\mathcal{E}, \mathcal{C}') \rightarrow RFun(\mathcal{E}, \mathcal{C})$$

functors which are right adjoints

Example: $\Omega^\infty: Sp(\mathcal{C}) \rightarrow \mathcal{C}$ exhibits $Sp(\mathcal{C})$ as a stable envelope of $\mathcal{C}$.

Def: A functor $p: X \rightarrow S$ of $\infty$-categories is a presentable fibration if it is both cartesian and cocartesian, and every fiber $X_s = X \times_S \{s\}$ is a presentable $\infty$-category.

Def: A stable envelope of a presentable fibration $p: \mathcal{C} \rightarrow \mathcal{D}$ is a functor $u: \mathcal{C}' \rightarrow \mathcal{C}$ st.

(i) $pou$ is a presentable fibration.

(ii) $u$ carries $(pou)$-cartesian morphisms of $\mathcal{C}'$ to $p$-cartesian morphisms of $\mathcal{C}$

(iii) $\forall D \in \mathcal{D}$, the induced map $\mathcal{C}'_D \rightarrow \mathcal{C}_D$ is a stable envelope of $\mathcal{C}'_D$.

Def: $\mathcal{C}$ presentable $\infty$-category. A tangent bundle to $\mathcal{C}$ is a functor $T\mathcal{C} \rightarrow Fun(\Delta^1, \mathcal{C})$
which exhibits $T_e$ as the stable envelope of the presentable fibration

$$\text{Fun}(\Delta^1, \mathcal{C}) \to \text{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}.$$ 

**Idea:** Objects of $T_e$ are pairs $(A, M)$, where $A \in \mathcal{C}, M \in \text{Sp}(\mathcal{C}/A)$.

For $\mathcal{C} = \text{CAlg}$, $M \in \text{Sp}(\text{CAlg}/A) \simeq \text{Mod}_A$.

The functor $T_e \to \text{Fun}(\Delta^1, \mathcal{C})$ sends $(A, M)$ to the projection $A \oplus M \to A$.

**Explicit construction of tangent bundle $T_e$:**

$$\text{Exc}(S^\infty_{(-)}, \mathcal{C}) \to \text{Fun}(\Delta^1, \mathcal{C})$$

$$(X : S^\infty_{(-)} \to \mathcal{C}) \mapsto (X(S^0) \to X(\ast))$$

**Def:** $\mathcal{C}$ presentable $\infty$-category. The *absolute cotangent complex functor* $L : \mathcal{C} \to T_e$ is a left adjoint to the composition $T_e \to \text{Fun}(\Delta^1, \mathcal{C}) \to \text{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}$.

**Rem:** relative adjunction $T_e \xrightarrow{G} \text{Fun}(\Delta^1, \mathcal{C})$

$$\begin{array}{c}
\downarrow \\
\mathcal{C}
\end{array}$$

**Rem:** For $A \in \mathcal{C}$, the object $L_A \in \text{Sp}(\mathcal{C}/A) \simeq (T_e)_A$ corresponds to the image of $\text{id}_A \in \mathcal{C}/A$ under the suspension spectrum functor $\Sigma^\infty_+ : \mathcal{C}/A \to \text{Sp}(\mathcal{C}/A)$.
1.3 The relative cotangent complex

\[ C \text{ presentable } \infty\text{-category}, \ A \in C \rightarrow \text{absolute cotangent complex} \quad \scriptstyle L_A \in \text{Sp}(C/A). \]

Goal: define a relative cotangent complex \( L_{B/A} \) associated to a morphism \( f: A \rightarrow B \) in \( C \).

Idea: Recall that for Kähler differentials, we have an exact sequence

\[
\Omega_A \otimes_B \Omega_B \rightarrow \Omega_B \rightarrow \Omega_{B/A} \rightarrow 0 \quad \text{for a homomorphism of rings.}
\]

So we want to define \( L_{B/A} \) via some cofiber sequence.

Def: \( C \) presentable \( \infty \)-category, \( p: T_C \rightarrow C \) tangent bundle.

A relative cofiber sequence in \( T_C \) is a pushout square in \( T_C \)

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
\text{Sp}(C/p(w)) & \rightarrow & Z
\end{array}
\]

st. each column lies in a fiber of \( p \).

Let \( E \subset \text{Fun}(\downarrow, T_C) \times \text{Fun}(\rightarrow, C) \times \text{Fun}(\leftarrow, C) \)

spanned by relative cofiber sequences.

The relative cotangent complex functor is the composition
\[ \text{Fun}(\Delta', C) \xrightarrow{l} \text{Fun}(\Delta', T_C) \xrightarrow{U} \mathcal{E} \xrightarrow{\epsilon} T_C \]

\[ \text{make relative \ take lower right corner} \]

cofiber sequence

\[(f: A \to B) \xrightarrow{\epsilon} L_{B/A} \in (T_C)_B \simeq \text{Sp}(C/B) \]

Rem: By definition, we have a relative cofiber sequence in \( T_C \)

\[ L_A \to L_B \]

\[ \downarrow \quad \downarrow \]

\[ 0 \to L_{B/A} \]

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cofiber sequence \( f: L_A \to L_B \to L_{B/A} \) in \( (T_C)_B \simeq \text{Sp}(C/B) \)

where \( f: \text{Sp}(C/A) \to \text{Sp}(C/B) \) denotes the functor induced by the cartesian fibration \( p \).

Example: • For \( f: A \to B \), \( A \) an initial object of \( C \), we have

\[ L_B \xrightarrow{\epsilon} L_{B/A} \in \text{Sp}(C/B). \]

• For \( f: A \simeq B \) an equivalence, we have \( L_{B/A} = 0 \in \text{Sp}(C/B) \).

Proposition: \( C \) presentable oo-category, \( T_C \) tangent bundle

\[ \begin{array}{ccc}
A & \xrightarrow{B} & C \\
\downarrow & & \downarrow \\
0 & \xrightarrow{L_{B/B}} & L_{C/B}
\end{array} \]

commutative diagram in \( C \)

⇒ pushout diagram \( L_{B/A} \to L_{C/A} \) in \( T_C \) (also a relative cofiber sequence.)
=> cofiber sequence $f: \mathcal{L}B/A \to \mathcal{L}C/A \to \mathcal{L}C/B$ in $\mathbf{Sp}(\mathcal{C}/c)$.

Proposition: Given a pushout diagram

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow f \\
A' & \rightarrow & B'
\end{array}
\]

we have an equivalence $f: \mathcal{L}B/A \sim \mathcal{L}B/A'$, i.e. $\mathcal{L}B/A \to \mathcal{L}B'/A'$ is a $p$-cocartesian morphism in $\mathbf{T}_{\mathcal{C}}$.

2. Deformation theory

2.1 Square-zero extensions

Recall: $R$ commutative ring. A square-zero extension of $R$ is a commutative ring $\tilde{R}$ with a surjection $\phi: \tilde{R} \to R$ s.t. $(\ker \phi)^2 = 0$. In this case, $M := \ker \phi$ inherits an $R$-module structure.

Def: $\mathcal{C}$ presentable $\infty$-category, $L: \mathcal{C} \to \mathbf{T}_{\mathcal{C}}$ a cotangent complex functor.

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{L} & T_{\mathcal{C}} \\
\downarrow & & \downarrow \\
\Delta' & = & \{ 0 \to 1 \}
\end{array}
\]

We call $\mathcal{M}^T(\mathcal{C})$ the tangent correspondence to $\mathcal{C}$.

It has a projection map $p: \mathcal{M}^T(\mathcal{C}) \to \Delta' \times \mathcal{C}$

Def: A derivation in $\mathcal{C}$ is a morphism $\eta: A \to M$ in $\mathcal{M}^T(\mathcal{C})$ where
A \in \mathcal{C}, \; M \in (\mathcal{T}_\mathcal{E})_A. \text{ By the cocartesian property, it can also be identified with a map } d: L_A \to M \text{ in the fiber } (\mathcal{T}_\mathcal{E})_A.

Let Der(\mathcal{C}) be the \infty\text{-category of derivations in } \mathcal{C}.

Def: \mathcal{C} \text{ presentable } \infty\text{-category. For every derivation } \eta: A \to M \text{ in } \mathcal{C}, form a pullback diagram } \begin{array}{ccc} A'^1 & \to & A \\ \downarrow & & \downarrow \eta \\ 0 & \to & M \end{array} \text{ in the } \omega\text{-cat } M^T(\mathcal{C}).

A morphism } f: A \to A \text{ in } \mathcal{C} \text{ is a square-zero extension if there exists a derivation } \eta: A \to M \text{ in } \mathcal{C} \text{ and an equiv } A \equiv A'^1 \text{ in } \mathcal{C}/A. \text{ We also call } A \text{ a square-zero extension of } A \text{ by } M[-1].