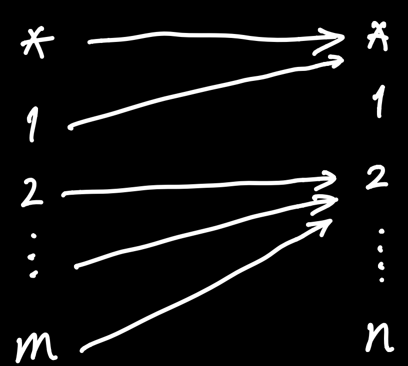
III. Derived Rings

1. ∞-Operads

Our language to make sense of commutativity and associativity up to homotopy.

Def: Segal's category $\mathcal{F}in_*$ of pointed finite sets:

$$\begin{cases} \text{Obj: } \langle n \rangle = \{*, 1, \cdots, n\}, & n \geq 0 \\ \text{Mor: } \alpha : \langle m \rangle \longrightarrow \langle n \rangle & \text{s.t. } \alpha(*) = * \end{cases}$$



$\forall\, 1 \leq i \leq n$, let $\rho^i : \langle n \rangle \longrightarrow \langle 1 \rangle$ sending $i \mapsto 1$, others $\mapsto *$.

A mor $f: \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}in_*$ is inert if $\forall\, i \in \langle n \rangle^\circ = \{1, \cdots, n\}$, $f^{-1}\{i\}$ has exactly one element.

Def: An ∞-operad is a functor $p: \mathcal{O}^\otimes \longrightarrow \mathcal{F}in_*$ between ∞-cats, s.t.

(1) $\forall$ inert mor $f: \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}in_*$ $\forall$ obj $C \in \mathcal{O}^\otimes_{\langle m \rangle}$, $\exists$ a p-cocartesian mor $\bar{f}: C \to C'$ in $\mathcal{O}^\otimes$ lifting f. Any such $\bar{f}$ is called inert. In particular, f induces a functor $f_! : \mathcal{O}^\otimes_{\langle m \rangle} \longrightarrow \mathcal{O}^\otimes_{\langle n \rangle}$

(2) let $C \in \mathcal{O}^{\otimes}_{\langle m \rangle}$, $C' \in \mathcal{O}^{\otimes}_{\langle n \rangle}$, $f : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}in_*$.

Let $\mathrm{Map}^{f}_{\mathcal{O}^{\otimes}}(C, C') \subset \mathrm{Map}_{\mathcal{O}^{\otimes}}(C, C')$ be union of conn comp lying over f

Choose p-cocartesian morphisms $C' \to C'_i$ lying over the inert morphisms $\rho^i : \langle n \rangle \to \langle 1 \rangle$ for $1 \le i \le n$. Then the induced map

$$\mathrm{Map}^{f}_{\mathcal{O}^{\otimes}}(C, C') \longrightarrow \prod_{1 \le i \le n} \mathrm{Map}^{\rho^i \circ f}_{\mathcal{O}^{\otimes}}(C, C'_i)$$

is a homotopy equivalence.

(3) $\forall n \geq 0$, the functors $\{ \rho^i_! : \mathcal{O}^{\otimes}_{\langle n \rangle} \to \mathcal{O} \}_{1 \le i \le n}$, $\mathcal{O} = \mathcal{O}_{\langle 1 \rangle}$, determine an equiv of ∞-cats $\phi : \mathcal{O}^{\otimes}_{\langle n \rangle} \to \mathcal{O}^{n}$.

Notation: $\mathcal{O} := \mathcal{O}^{\otimes}_{\langle 1 \rangle}$ is called the underlying ∞-category of $\mathcal{O}^{\otimes}$.

Given $X_1, \cdots, X_n, Y \in \mathcal{O}$, let $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{1 \le i \le n}, Y)$ denote the union of components of $\mathrm{Map}_{\mathcal{O}^{\otimes}}(\underbrace{X_1 \oplus \cdots \oplus X_n}_{\in \, \mathcal{O}^{\otimes}_{\langle n \rangle}}, Y)$ which lies over the unique mor $\beta : \langle n \rangle \to \langle 1 \rangle$ s.t. $\beta^{-1}\{*\} = \{*\}$.

$$\beta \quad \begin{array}{ccc} * & \longrightarrow & * \\ 1 & \longrightarrow & 1 \\ \vdots & \nearrow & \\ n & \nearrow & \end{array}$$

Idea: We think of an ∞-operad as a category $\mathcal{O}$ together with "multi-morphisms" $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{1 \le i \le n}, Y)$ whose compositions are associative up to homotopy.

Example: The commutative ∞-operad $\mathrm{Comm}^{\otimes} := \mathcal{F}in_*$

Def: $\mathcal{O}^{\otimes}$, $\mathcal{O}'^{\otimes}$ ∞-operads. An ∞-operad map from $\mathcal{O}^{\otimes}$ to $\mathcal{O}'^{\otimes}$ is a functor $f: \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ of ∞-cats over $\mathcal{F}in_*$ which carries inert mor in $\mathcal{O}^{\otimes}$ to inert mor in $\mathcal{O}'^{\otimes}$.

Let $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ denote the full subcat of $\mathrm{Fun}_{\mathcal{F}in_*}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes})$ spanned by ∞-operad maps.

$$\begin{tikzcd} \mathcal{O} \arrow[dr] & & \mathcal{O}' \arrow[dl] \\ & \mathcal{O}'' \arrow[d] & \\ & \mathcal{F}in_* & \end{tikzcd}$$

$\mathrm{Alg}_{\mathcal{O}/\mathcal{O}''}(\mathcal{O}')$

$\mathrm{Alg}_{/\mathcal{O}}(\mathcal{O}') := \mathrm{Alg}_{\mathcal{O}/\mathcal{O}}(\mathcal{O}')$

where $\mathcal{O} = \mathcal{O}''$.

Def: $\mathcal{O}^{\otimes}$ ∞-operad. A cocartesian fibration $p: \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is called a cocartesian fibration of ∞-operads if the composition $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \to \mathcal{F}in_*$ is an ∞-operad. In this case, $\mathcal{C}$ is called an $\mathcal{O}$-monoidal ∞-category.

Idea: $\forall f \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{1 \le i \le n}, Y)$, the cocartesian fibration p determines a functor

$$\otimes_f : \prod_{1 \le i \le n} \mathcal{C}_{X_i} \longrightarrow \mathcal{C}_Y$$

fiber of $\mathcal{C}$ at X_i

Example: A symmetric monoidal ∞-cat $\mathcal{C}$ is a $\mathcal{F}in_*$-monoid ∞-cat, i.e. a cocartesian fibration $\mathcal{C}^{\otimes} \to \mathcal{F}in_*$ s.t. $\forall n \geq 0$, the maps $\{\rho^i : \langle n \rangle \to \langle 1 \rangle\}_{1 \le i \le n}$ induces functors $\rho^i_! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\langle 1 \rangle}$ which determine an equivalence $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^n$.

The morphisms $\alpha : \langle 0 \rangle \to \langle 1 \rangle$ and $\beta : \langle 2 \rangle \to \langle 1 \rangle$ in $\mathcal{F}in_*$ determines

$\mathrm{CAlg}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{F}in_*}(\mathcal{C})$ ∞-category of commutative algebra objects of $\mathcal{C}$.

functors $\quad \Delta^{\circ} \longrightarrow \mathcal{C} \qquad \mathcal{C}\times\mathcal{C} \to \mathcal{C}$

Corresponding to the unit object $1 \in \mathcal{C}$ and the tensor product.

Def:
$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[dr, "p"'] & & \mathcal{D}^{\otimes} \arrow[dl, "q"] \\ & \mathcal{O}^{\otimes} & \end{tikzcd}$$
$\mathcal{O}$-monoidal ∞-categories

$f \in \mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$ is a *$\mathcal{O}$-monoidal functor* if it carries p-cocartesian mor to q-cocartesian mor.

In the special case $\mathcal{O}^{\otimes} = \mathcal{F}in_*$, we say *symmetric monoidal functor*.

Def: $\mathcal{C}$ ∞-category. A symmetric monoidal structure on $\mathcal{C}$ is *cartesian* if

- the unit obj $1_{\mathcal{C}} \in \mathcal{C}$ is final
- $\forall C, D \in \mathcal{C}$, the canonical maps $C \simeq C \otimes 1_{\mathcal{C}} \leftarrow C \otimes D \to 1_{\mathcal{C}} \otimes D \simeq D$

exhibit $C \otimes D$ as a product of C and D in the ∞-cat $\mathcal{C}$.

Dually $\rightsquigarrow$ cocartesian symmetric monoidal structure.

HA 2.4.1: $\mathcal{C}$ ∞-cat admitting finite products. Then $\mathcal{C}$ admits a cartesian symmetric monoidal structure, unique up to equivalence.

In this case, the ∞-cat $\mathrm{CAlg}(\mathcal{C})$ admits a direct description in terms of monoids.

2. Algebras and modules

$\mathcal{C}^{\otimes}$ symmetric monoidal ∞-cat $\qquad \mathcal{O} := \mathrm{Fin}_*$

$\mathrm{CAlg}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$

Evaluate at $\langle 1 \rangle \in \mathrm{Fin}_* \rightsquigarrow$ Forgetful functor $\theta : \mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$

HA3.1.3.14: Assume that the underlying ∞-cat $\mathcal{C}$ admits countable colimits, and $\otimes$ preserves countable colimits. Then θ admits a left adjoint, which is given informally by the formula

$$C \longmapsto \mathrm{Sym}^* C = \coprod_n C^{\otimes n}/\Sigma_n$$

"∞-categorical quotient by the symmetric group"

HA3.2: 1) If $\mathcal{C}$ is complete (i.e. has small limits), then so is $\mathrm{CAlg}(\mathcal{C})$.

The forgetful functor θ detects (i.e. preserves and reflects) limits.

2) If $\mathcal{C}$ is cocomplete, and $\otimes$ is compatible with colimits, then $\mathrm{CAlg}(\mathcal{C})$ is cocomplete. The forgetful functor θ detects sifted colimits, but not general colimits.

(A simplicial set K is sifted if it is nonempty, and the diagonal map $K \to K \times K$ is cofinal.

Idea: sifted $\supset$ filtered + quotient by equiv relations.)

3) If $\mathcal{C}$ is presentable, and $\otimes$ is compatible with colimits, then $\mathrm{CAlg}(\mathcal{C})$ is presentable.

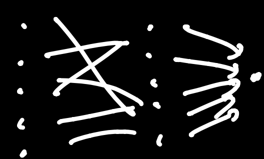
4) $\exists$ a cocartesian symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ s.t. the forgetful functor is monoidal. (Informally, the tensor product on $\mathcal{C}$ induces a tensor product on comm alg. objects)

Def: Associative operad Assoc

It is a colored operad having a single object $\mathfrak{a}$

$\forall$ finite set I, the set of operations $\mathrm{Mul}_{\mathrm{Assoc}}(\{\mathfrak{a}\}_{i\in I}, \mathfrak{a})$ is the set of linear orderings on I.

I



Composition of linear orderings

$\rightsquigarrow$ ∞-operad $\mathrm{Assoc}^{\otimes} \to \mathrm{Fin}_*$ called associative ∞-operad

$\mathcal{C}^{\otimes}$ ∞-operad equipped with a fibration $q: \mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$

$\mathrm{Alg}(\mathcal{C}) := \mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C})$ ∞-operad sections of q, called the ∞-cat of associative algebra objects of $\mathcal{C}$.

A monoidal ∞-cat is a cocartesian fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$

HA 4.1.1.14: Let $p: \mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ ∞-operad. Then $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-cat if and only if the induced map $p': \mathrm{Assoc}^{\otimes} \times_{\mathrm{Fin}_*} \mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$

is a monoidal ∞-cat.

Def: Left module operad LM.

It is a colored operad having two elements $\mathfrak{a}$ and $\mathfrak{m}$

$\forall$ objects $\{X_i\}_{i\in I}$ and Y of LM

$$\mathrm{Mul}_{LM}(\{X_i\}_{i\in I}, Y) := \begin{cases} \text{If } Y=\mathfrak{a},\ X_i=\mathfrak{a}\ \forall i, \text{ then it is the collection of all linear orderings on } I; \\ \text{If } Y=\mathfrak{m}, \text{ and exactly one } X_i=\mathfrak{m}, \text{ then it is the collection of linear orderings on } I \text{ s.t. the last } X_i=\mathfrak{m}. \\ \text{In all other cases, it is } \phi. \end{cases}$$

$\rightsquigarrow$ ∞-operad $\mathcal{LM}^{\otimes}$

We have natural $\mathrm{Assoc}^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes} \xrightarrow{\text{fibration}} \mathrm{Assoc}$

$\mathcal{C}^{\otimes} \longrightarrow \mathrm{Assoc}^{\otimes}$ fibration of ∞-operads

∞-category of left modules $\mathrm{LMod}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{LM}/\mathrm{Assoc}}(\mathcal{C})$

For $A \in \mathrm{Alg}(\mathcal{C})$, we define ∞-cat of left A-modules

$$\mathrm{LMod}_A(\mathcal{C}) := \mathrm{LMod}(\mathcal{C}) \underset{\mathrm{Alg}(\mathcal{C})}{\times} \{A\}.$$

HA 4.2: $\mathcal{C}$ monoidal ∞-cat, $A \in \mathrm{Alg}(\mathcal{C})$, forgetful functor

$$\theta: \mathrm{LMod}_A(\mathcal{C}) \longrightarrow \mathcal{C}.$$

(1) θ has a left adjoint, given by "free module".

(2) If $\mathcal{C}$ is complete, then so is $LMod_A(\mathcal{C})$, and θ detects limits.

(3) If $\mathcal{C}$ is cocomplete, and $\otimes$ preserves colimits, then $LMod_A(\mathcal{C})$ is cocomplete, and θ detects colimits.

(4) If $\mathcal{C}$ is presentable, and $\otimes$ preserves colimits, then $LMod_A(\mathcal{C})$ is presentable.

(5) Assume $\mathcal{C}$ is symmetric monoidal, admits simplicial colimits, and $\otimes$ preserves simplicial colimits. $A \in CAlg(\mathcal{C})$.

Then $Mod_A(\mathcal{C}) := LMod_A(\mathcal{C})$ is a symmetric monoidal ∞-cat, and the forgetful functor θ is lax-monoidal (i.e. only a map of ∞-operads).