Adjoint functors:

Def: Consider cartesian fibration

$$\begin{array}{ccc} M & \mathcal{C} & \mathcal{D} \\ {\scriptstyle p}\downarrow & & \\ \Delta^1 & 0 & \longrightarrow\ 1 \end{array}$$

$\xrightarrow{\text{straightening}}$ functor $g: \mathcal{D} \to \mathcal{C}$

Def: $\mathcal{C}, \mathcal{D}$ ∞-cats. An adjunction between $\mathcal{C}$ and $\mathcal{D}$ is a functor $q: M \to \Delta^1$ which is both a cartesian fibration and a cocartesian fibration together with equivalences $\mathcal{C} \simeq M_0$ and $\mathcal{D} \simeq M_1$.

Let $f: \mathcal{C} \to \mathcal{D}$ and $g: \mathcal{D} \to \mathcal{C}$ be functors associated to M.

We say that f is left adjoint to g, and g is right adjoint to f.

HTT 5.2.2.8: $\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$ ∞-cats. TFAE:

- f is left adjoint to g
- $\exists$ a unit transformation $u: id_{\mathcal{C}} \to g \circ f$, i.e. $\forall C \in \mathcal{C}, D \in \mathcal{D}$,

the composition $Map_{\mathcal{D}}(f(C), D) \longrightarrow Map_{\mathcal{C}}(g(f(C)), g(D)) \xrightarrow{u(C)} Map_{\mathcal{C}}(C, g(D))$ is a homotopy equivalence.

HTT 5.2.3.5: $\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$ adjunction. Then

f preserves all colimits which exist in $\mathcal{C}$

g preserves all limits which exist in $\mathcal{D}$.

Proposition: $f: \mathcal{C} \to \mathcal{D}$ $\quad \mathcal{E}$ $\quad\infty$-cats

Composition with f gives a functor $f^*: \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E})$.

The left (right) adjoint to f^* is the left (right) Kan extension functor.

These adjoints exist if and only if every functor $\mathcal{C} \to \mathcal{E}$ admits a Kan extension.

Def: A full subcat $\mathcal{C}_0 \subset \mathcal{C}$ is a **localization** of $\mathcal{C}$ if the inclusion has a left adjoint.

Rem: $\mathcal{C}_0 \underset{i}{\overset{L}{\rightleftarrows}} \mathcal{C}$ The composition $\mathcal{C}_0 \hookrightarrow \mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$ is an equiv of ∞-cats.

Collection of mor α s.t. $L(\alpha)$ is an equiv

Def: An ∞-cat is **presentable** if it is accessible and admits small colimits.

set-theoretical condition

HTT 5.5.2.4: A presentable ∞-cat admits small limits.

HTT 5.5.1.1 (Simpson): An ∞-cat is presentable if and only if it arises as an (accessible) localization of an ∞-cat of presheaves.

Adjoint functor theorem (HTT 5.5.2.9):

Let $F: \mathcal{C} \to \mathcal{D}$ a functor between presentable ∞-cats:

(1) F has a right adjoint if and only if it preserves small colimits.

(2) F has a left adjoint if and only if it preserves small limits and κ-filtered colimits for some regular cardinal κ.

HTT 5.5.4.15: $\mathcal{C}$ presentable ∞-cat. Then

(accessible) localizations of $\mathcal{C}$ $\Longleftrightarrow$ inverting collection of morphisms that are strongly saturated and of small generation.

8. Stable ∞-categories "linearized ∞-cats for doing algebra"

Def: An ∞-cat $\mathcal{C}$ is stable if

(1) $\exists$ a zero obj $0 \in \mathcal{C}$, i.e. both initial and final

(2) Every mor g in $\mathcal{C}$ admits a fiber and a cofiber, i.e.

$\exists$ pullback
$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ 0 \arrow[r] & Y \end{tikzcd}$$
W: fiber of g

$\exists$ pushout
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$
Z : cofiber of g

(3) A triangle in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence, i.e.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$
triangle

is a $\underbrace{\text{pullback}}_{\text{fiber seq.}}$ if and only if it is a $\underbrace{\text{pushout}}_{\text{cofiber seq}}$.

Def: $\mathcal{C}$ stable ∞-cat. $X \in \mathcal{C}$. Form fiber/cofiber seq:

$X \longrightarrow 0$ $\qquad\qquad$ $\Omega X \longrightarrow 0$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array} \qquad\qquad \begin{array}{ccc} \downarrow & & \downarrow \\ 0 & \longrightarrow & X \end{array}$$

Suspension loop

HTT4.3.2.15 $\Rightarrow$ $\exists$ suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$

loop functor $\Omega : \mathcal{C} \to \mathcal{C}$

They are mutually inverse equivalences.

For $n \geq 0$, $X[n] :=$ n^{th} power of suspension

$n \leq 0$, $X[n] := (-n)^{th}$ power of loop

Rem: $\mathcal{C}$ stable ∞-cat, $f: X \to Y$ in $\mathcal{C}$. Form pullbacks

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r, "h"] & X[1] \end{tikzcd}$$

Then the image of $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in the homotopy cat $h\mathcal{C}$ is called a distinguished triangle. The collection of distinguished triangles endow $h\mathcal{C}$ with the classical structure of a triangulated cat.

HA1.1.3.4: A stable ∞-cat admits all finite limits and colimits, and the pushout squares in $\mathcal{C}$ coincide with pullback squares in general.

Equivalent definitions of stable ∞-cats:

HA1.4.2.27: Let $\mathcal{C}$ be a pointed ∞-cat (i.e. has a 0-obj). TFAE:

(1) $\mathcal{C}$ is stable

(2) $\mathcal{C}$ admits finite colimits, and the suspenoion functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is

an equiv.

(3) $\mathcal{C}$ admits finite limits, and the loop functor $\Omega: \mathcal{C} \to \mathcal{C}$ is an equiv.

HA 1.1.4.1: A functor $F: \mathcal{C} \to \mathcal{D}$ between stable ∞-cats is called exact if and only if the following equiv cond hold:

(1) F preserves fiber sequences

(2) F is left exact, i.e. commutes with finite limits

(3) F is right exact, i.e. commutes with finite colimits.

HA 1.1.4.4, 1.1.4.6: Let $\mathcal{C}at_\infty^{Ex} \subset \mathcal{C}at_\infty$ be the sub cat spanned by stable ∞-cats and exact functors. Then $\mathcal{C}at_\infty^{Ex}$ admits small limits and small filtered colimits, and they are preserved by the inclusion.

Def: A t-structure on a stable ∞-cat $\mathcal{C}$ is a pair of full subcats $\mathcal{C}_{\geq 0}$, $\mathcal{C}_{\leq 0}$ s.t.

(1) $\forall X \in \mathcal{C}_{\geq 0}$, $Y \in \mathcal{C}_{\leq 0}$, $\pi_0 \operatorname{Map}(X, Y[-1]) = 0$

(2) $\mathcal{C}_{\geq 1} \subset \mathcal{C}_{\geq 0}$, $\mathcal{C}_{\leq -1} \subset \mathcal{C}_{\leq 0}$, where $\mathcal{C}_{\geq n} := \mathcal{C}_{\geq 0}[n]$

$\mathcal{C}_{\leq n} := \mathcal{C}_{\leq 0}[n]$

(3) $\forall X \in \mathcal{C}$, $\exists$ fiber seq $X' \to X \to X''$ where $X' \in \mathcal{C}_{\geq 0}$, $X'' \in \mathcal{C}_{\leq -1}$

HA 1.2.1.5: $\forall n \in \mathbb{Z}$, $\mathcal{C}_{\leq n} \subset \mathcal{C}$ is a localization $\rightsquigarrow$ left adjoint $\tau_{\leq n}$ truncation

Dually, $\mathcal{C}_{\geq n} \subset \mathcal{C}$ has a right adjoint $\tau_{\geq n}$.

Def: The heart $\mathcal{C}^\heartsuit :=$ full subcat $\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0} \subset \mathcal{C}$

$$\pi_0 := \tau_{\leq 0} \circ \tau_{\geq 0} \simeq \tau_{\geq 0} \circ \tau_{\leq 0} : \mathcal{C} \longrightarrow \mathcal{C}^{\heartsuit}$$

$\forall n \in \mathbb{Z}$, $\pi_n := \pi_0 \circ [-n]$.

Rem: For $X, Y \in \mathcal{C}^{\heartsuit}$, $\pi_n \mathrm{Map}(X, Y) \simeq \mathrm{Map}(X, Y[-n]) = 0$ for $n > 0$.

So $\mathcal{C}^{\heartsuit}$ is a 1-category, it is in fact an abelian category.

Example: $\mathcal{A}$ Grothendieck abelian cat

$\mathrm{Ch}(\mathcal{A})$: the cat of chain complexes

$\mathcal{D}(\mathcal{A}) := \mathrm{Ch}(\mathcal{A})[\mathrm{qis}^{-1}]$, inverting all quasi-isomorphisms,

called the derived ∞-category.

HA 1.3.5: $\mathcal{D}(\mathcal{A})$ is a presentable stable ∞-category.

It has a t-structure $(\mathcal{D}(\mathcal{A})_{\geq 0}, \mathcal{D}(\mathcal{A})_{\leq 0})$

spanned by chain complexes M with $H_i(M) = 0$ for $i < 0$.

Proof: Use dg structure and model structure to study the localization.

Notation: Right bounded derived ∞-cat $\mathcal{D}^-(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$ spanned by chain complexes M with $H_i(M) = 0$ for $i \ll 0$.

Left bounded .... $\mathcal{D}^+(\mathcal{A})$ .....

Classically, $\mathcal{D}^-(\mathcal{A})$ is constructed from projective resolutions

$\mathcal{D}^+(\mathcal{A})$ injective

9. Spectra and stabilization

Want: construct stable ∞-cats from ∞-cats admitting finite limits,

or "linearize" ∞-cats.

Idea: formally invert the loop functor

Def: $\mathcal{C}$ ∞-cat admitting finite limits. Let $\mathcal{C}_* := \mathcal{C}_{*/}$ the ∞-cat of pointed objects of $\mathcal{C}$. The ∞-cat of spectrum objects of $\mathcal{C}$:

$$Sp(\mathcal{C}) := \lim \left(\cdots \to \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \right)$$

Def: ∞-cat of finite spaces $\mathcal{S}^{fin} \subset \mathcal{S}$ full subcat generated by $*$ under finite colimits.

$\mathcal{C}$ ∞-cat admitting finite limits.

Equivalently, a spectrum object of $\mathcal{C}$ is a reduced, excisive functor

$X : \mathcal{S}^{fin}_* \to \mathcal{C}$ $\qquad$ $X(*) = *$ $\qquad$ takes pushout to pullbacks

$$Sp(\mathcal{C}) = Exc_*(\mathcal{S}^{fin}_*, \mathcal{C}) \subset Fun(\mathcal{S}^{fin}_*, \mathcal{C})$$ full subcat spanned by spectrum objects.

HA 1.4.2.17: $Sp(\mathcal{C})$ is a stable ∞-cat.

Notation: $\Omega^\infty : Sp(\mathcal{C}) \to \mathcal{C}$ evaluation at the 0-sphere S^0.

$\forall n \in \mathbb{Z}$, $\Omega^{\infty-n} := \Omega^\infty \circ [n]$.

Adjoint functor theorem $\Rightarrow$ Ω^∞ has left adjoint $\Sigma^\infty_+ : \mathcal{C} \to Sp(\mathcal{C})$

Next we specialize to the case $\mathcal{C} = \mathcal{S}$.

Def: A spectrum is a spectrum object of the ∞-cat of spaces $\mathcal{S}$.

The ∞-cat of spectra $Sp := Sp(\mathcal{S}_*)$

Sphere spectrum $\mathbb{S} := \Sigma^\infty_+(*) \in \mathrm{Sp}$

More concretely, a spectrum is a sequence of pointed spaces $\{X_n\}_{\geq 0}$ equipped with homotopy equivalences $X_n \xrightarrow{\sim} \Omega X_{n+1}$.

HA 1.4.3.6: The full subcat $(\mathrm{Sp})_{\leq -1} \subset \mathrm{Sp}$ spanned by objects X s.t. $\Omega^\infty(X) \in \mathcal{S}$ is contractible.

$\rightsquigarrow$ t-structure on Sp, whose heart $\mathrm{Sp}^\heartsuit \simeq \mathrm{Ab}$, the cat of abelian group.