Adjoint functors:

Definition: Consider cartesian fibration \[
\begin{array}{ccc}
M & \xrightarrow{f} & C \\
p \downarrow & & \downarrow \\
\Delta' & \rightarrow & 0 \rightarrow 1
\end{array}
\]

The straightening \[
\text{straightening} \xrightarrow{g} \text{functor } g: D \rightarrow C
\]

Definition: \(\mathcal{C}, \mathcal{D}\) ∞-cats. An adjunction between \(\mathcal{C}\) and \(\mathcal{D}\) is a functor \(g: M \rightarrow \Delta'\) which is both a cartesian fibration and a cocartesian fibration together with equivalences \(\mathcal{C} \simeq M_0\) and \(\mathcal{D} \simeq M_1\).

Let \(f: \mathcal{C} \rightarrow \mathcal{D}\) and \(g: \mathcal{D} \rightarrow \mathcal{C}\) be functors associated to \(M\). We say that \(f\) is left adjoint to \(g\), and \(g\) is right adjoint to \(f\).

HTT 5.2.2.8: \(\mathcal{C} \xleftarrow{f} \xrightarrow{g} \mathcal{D}\) ∞-cats. TFAE:

- \(f\) is left adjoint to \(g\).
- \(\exists\) a unit transformation \(\eta: \text{id}_{\mathcal{C}} \rightarrow g \circ f\), i.e. \(\forall C \in \mathcal{C}, D \in \mathcal{D}\), the composition \(\text{Map}_D(f(C), D) \rightarrow \text{Map}_C(g(f(C)), g(D)) \xrightarrow{\eta(C)} \text{Map}_C(C, g(D))\) is a homotopy equivalence.

HTT 5.2.3.5: \(\mathcal{C} \xleftarrow{f} \xrightarrow{g} \mathcal{D}\) adjunction. Then

- \(f\) preserves all colimits which exist in \(\mathcal{C}\)
- \(g\) preserves all limits which exist in \(\mathcal{D}\).
Proposition: \( f: C \to D \) \( E \) \( \infty \)-cats

Composition with \( f \) gives a functor \( f^*: \text{Fun}(D, E) \to \text{Fun}(C, E) \).

The left (right) adjoint to \( f^* \) is the left (right) Kan extension functor.

These adjoints exist if and only if every functor \( C \to E \) admits a Kan extension.

Def: A full subcat \( C_0 \subseteq C \) is a localization of \( C \) if the inclusion has a left adjoint.

Rem: \( C_0 \leftarrow i \rightarrow C \)

The composition \( C_0 \leftarrow C \to C[W^{-1}] \)

is an equiv of \( \infty \)-cats.

Def: An \( \infty \)-cat is presentable if it is accessible and admits small colimits.

\textsc{Htt}5.5.2.4: A presentable \( \infty \)-cat admits small limits.

\textsc{Htt}5.5.1.1 (Simpson): An \( \infty \)-cat is presentable if and only if it arises as an (accessible) localization of an \( \infty \)-cat of presheaves.

Adjoint functor theorem (\textsc{Htt}5.5.2.9):

Let \( F: C \to D \) a functor between presentable \( \infty \)-cats:

1. \( F \) has a right adjoint if and only if it preserves small colimits.
2. \( F \) has a left adjoint if and only if it preserves small limits and \( \kappa \)-filtered colimits for some regular cardinal \( \kappa \).
HTT 5.4.15: \( \mathcal{C} \) presentable \( \infty \)-cat. Then

(\text{accessible})\ localizations\ of\ \( \mathcal{C} \) \iff \text{inverting collection of morphisms that are strongly saturated and of small generation.}

\( \mathbf{8.} \) Stable \( \infty \)-categories “linearized \( \infty \)-cats for doing algebra”

\textbf{Def:} An \( \infty \)-cat \( \mathcal{C} \) is \textit{stable} if

1. \( \exists \) a zero \( \text{obj} \) \( 0 \in \mathcal{C} \), i.e. both initial and final

2. Every \( \text{mor} \ g \) in \( \mathcal{C} \) admits a fiber and a cofiber, i.e.

\[\begin{array}{ccc}
\exists \text{ pullback } & W & \rightarrow X & \exists \text{ pushout } & X & \rightarrow Y \\
& \downarrow & g & \downarrow & \downarrow & \downarrow \\
& 0 & \rightarrow Y & 0 & \rightarrow Z
\end{array}\]

\( W \): fiber of \( g \) \hspace{1cm} \( Z \): cofiber of \( g \)

3. A triangle in \( \mathcal{C} \) is a fiber sequence if and only if it is a cofiber sequence, i.e.

\[\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
0 & \rightarrow & Z
\end{array}\]

\( \triangle \) is a pullback if and only if it is a pushout.

\textbf{Def:} \( \mathcal{C} \) stable \( \infty \)-cat. \( X \in \mathcal{C} \). Form fiber/cofiber seq:

\[X \rightarrow 0 \quad \Omega X \rightarrow 0\]
They are mutually inverse equivalences.

For \( n \geq 0 \), \( X[n] := n^{\text{th}} \text{ power of suspension} \)
\[ X \to \Sigma^n X \]
For \( n \leq 0 \), \( X[n] := (-n)^{\text{th}} \text{ power of loop} \)
\[ X \to \Omega^n X \]

Rem: \( C \) stable \( \infty \)-cat, \( f \colon X \to Y \) in \( C \). Form pullbacks
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow g \\
0 & \xrightarrow{h} & Z \\
\end{array}
\]
Then the image of \( X \to Y \to Z \to X[1] \) in the homotopy cat \( hC \) is called a distinguished triangle. The collection of distinguished triangles endow \( hC \) with the classical structure of a triangulated cat.

HA1.1.3.4: A stable \( \infty \)-cat admits all finite limits and colimits, and the pushout squares in \( C \) coincide with pullback squares in general.

Equivalent definitions of stable \( \infty \)-cats:

HA1.4.2.27: let \( C \) be a pointed \( \infty \)-cat (i.e. has a 0-obj). TFAE:
1. \( C \) is stable
2. \( C \) admits finite colimits, and the suspension functor \( \Sigma : C \to C \) is
(3) \( C \) admits finite limits, and the loop functor \( \Omega: C \to C \) is an equiv.

HA 1.1.4.1: A functor \( F: C \to D \) between stable \( \infty \)-cats is called exact if and only if the following equiv cond hold:
(1) \( F \) preserves fiber sequences
(2) \( F \) is left exact, i.e. commutes with finite limits
(3) \( F \) is right exact, i.e. commutes with finite colimits.

HA 1.1.4.4, 1.1.4.6: let \( \text{Cat}^\text{ex}_\infty \subset \text{Cat}_\infty \) be the sub cat spanned by stable \( \infty \)-cats and exact functors. Then \( \text{Cat}^\text{ex}_\infty \) admits small limits and small filtered colimits, and they are preserved by the inclusion.

Def: A \( t \)-structure on a stable \( \infty \)-cat \( C \) is a pair of full subcats \( \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0} \) s.t.
(1) \( \forall X \in \mathcal{C}_{\geq 0}, Y \in \mathcal{C}_{\leq 0}, \pi_0 \text{Map} (X, Y(-1)) = 0 \)
(2) \( \mathcal{C}_{\geq 1} \subset \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq -1} \subset \mathcal{C}_{\leq 0}, \) where \( \mathcal{C}_{\geq n} := \mathcal{C}_{\geq 0}[n] \)
\( \mathcal{C}_{\leq n} := \mathcal{C}_{\leq 0}[n] \)
(3) \( \forall X \in C, \exists \) fiber seq \( X' \to X \to X'' \) where \( X' \in \mathcal{C}_{\geq 0}, X'' \in \mathcal{C}_{\leq -1} \)

HA 1.2.1.5: \( \forall n \in \mathbb{Z}, \mathcal{C}_{\leq n} \subset C \) is a localization \( \implies \) left adjoint \( \tau \leq n \) truncation
Dually, \( \mathcal{C}_{\geq n} \subset C \) has a right adjoint \( \tau \geq n \).

Def: The heart \( C^\heartsuit := \) full subcat \( \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0} \subset C \)
\[
\pi_0 := \mathcal{I}_{\leq 0} \cdot \mathcal{I}_{\geq 0} \cong \mathcal{I}_{\geq 0} \cdot \mathcal{I}_{\leq 0} : \mathcal{C} \rightarrow \mathcal{C}^\heartsuit
\]

\forall n \in \mathbb{Z}, \quad \pi_n := \pi_0 \cdot [-n].

Rem: For \( X, Y \in \mathcal{C}^\heartsuit \), \( \pi_n \text{Map}(X,Y) \cong \text{Map}(X,Y[-n]) = 0 \) for \( n > 0 \).

So \( \mathcal{C}^\heartsuit \) is a I-category, it is in fact an abelian category.

Example: A Grothendieck abelian cat

\( \text{Ch}(A) \): the cat of chain complexes

\( \text{D}(A) := \text{Ch}(A)[\text{qis}^{-1}] \), inverting all quasi-isomorphisms, called the derived \( \infty \)-category.

HA 1.35: \( \text{D}(A) \) is a presentable stable \( \infty \)-category.

It has a t-structure \((\text{D}(A)_{\geq 0}, \text{D}(A)_{\leq 0})\)

\[ \text{spanned by chain complexes } M \text{ with } H_i(M) = 0 \text{ for } i < 0. \]

Proof: Use dg structure and model structure to study the localization.

Notation: Right bounded derived \( \infty \)-cat \( D^-(A) \subset D(A) \) spanned by chain complexes \( M \) with \( H_i(M) = 0 \) for \( i < 0. \)

Left bounded \( \cdots \quad D^+(A) \quad \cdots \)

Classically, \( D^-(A) \) is constructed from projective resolutions

\( D^+(A) \) injective

9. Spectra and stabilization

Want: construct stable \( \infty \)-cats from \( \infty \)-cats admitting finite limits, or “linearize” \( \infty \)-cats.
Idea: formally invert the loop functor

**Def:** $\infty$-$\text{cat}$ admitting finite limits. Let $E_* := E_{/\ast}$ the $\infty$-$\text{cat}$ of pointed objects of $E$. The $\infty$-$\text{cat}$ of spectrum objects of $E$:

$$Sp(E) := \lim \left( \cdots \rightarrow E_{/\ast} \xrightarrow{\Omega} E_{/\ast} \xrightarrow{\Omega} E_{/\ast} \right)$$

**Def:** $\infty$-$\text{cat}$ of finite spaces $S_{\text{fin}} \subset S$ full subcat generated by $\ast$ under finite colimits.

$E$ $\infty$-$\text{cat}$ admitting finite limits.

Equivalently, a spectrum object of $E$ is a reduced, excisive functor $X : S_{\ast}^{\text{fin}} \rightarrow E$

$$Sp(E) = \text{Exc}_*(S_{\ast}^{\text{fin}}, E) \subset \text{Fun}(S_{\ast}^{\text{fin}}, E)$$ full subcat spanned by spectrum objects.

**HA 1.4.2.17:** $Sp(E)$ is a stable $\infty$-$\text{cat}$.

**Notation:** $Q^\infty : Sp(E) \rightarrow C$ evaluation at the 0-sphere $S^0$

$$\forall n \in \mathbb{Z}, \quad Q^{\infty-n} := Q^{\infty} \circ [n]$$

Adjoint functor theorem $\Rightarrow Q^\infty$ has left adjoint $\Sigma^\infty : C \rightarrow Sp(E)$

Next we specialize to the case $E = S$.

**Def:** A spectrum is a spectrum object of the $\infty$-$\text{cat}$ of spaces $S$.

The $\infty$-$\text{cat}$ of spectra $Sp := Sp(S_{/\ast})$
Sphere spectrum \( S := \Sigma^\infty_+(\ast) \in Sp \)

More concretely, a spectrum is a sequence of pointed spaces \( \{X_n\}_{n \geq 0} \)
equipped with homotopy equivalences \( X_n \sim \Omega \Sigma X_{n+1} \).

HA 14.3.6: The full subcat \((Sp)_{\leq 1} \subset Sp\) spanned by objects \( X \)
s.t. \( \Omega^\infty(X) \in S \) is contractible.

\( \implies \) t-structure on \( Sp \), whose heart \( Sp^0 \cong Ab \), the cat of abelian group.