I.6 Deformation theory

Recall: X smooth algebraic variety / $\mathbb{C}$

tangent bundle $T_X := (\Omega^1_X)^\vee$

(a) $\forall$ 1^{st}-order deformation X_1 of X,

$\{$Automorphisms of X_1 that restrict to id on $X\} \simeq H^0(X, T_X)$

(b) $\{$Isom classes of 1^{st}-order deformations of $X_0\} \simeq H^1(X, T_X)$

Q: Beyond the smooth case?

A: Use cotangent complex instead of Kähler differentials.

Replace $H^0(X, T_X)$ by $\mathrm{Hom}(L_X, \mathcal{O}_X)$

$H^1(X, T_X)$ by $\mathrm{Ext}^1(L_X, \mathcal{O}_X)$

Note that $H^0(L_X) \simeq \Omega^1_X$, and $L_X \simeq \Omega^1_X$ if X is smooth.

Quot schemes: X proj variety / $\mathbb{C}$ $\quad \mathcal{F}$ quasi-coherent sheaf on X

We have the quot scheme Quot constructed by Grothendieck classifying quotients of $\mathcal{F}$, or exact sequences $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$

We want compute the cotangent complex of Quot at a point $[\mathcal{F}'']$.

The 0^{th}-cohomology $=$ Zariski cotangent space $= \mathrm{Hom}(\mathcal{F}', \mathcal{F}'')$.

Higher cohomologies are very difficult to compute because Quot is usually singular.

Moral: The cotangent complex of Quot is not the right object to study. We should instead study the derived enhancement

Quot^{+} of Quot, whose cotangent complex at a point $[F'']$ will be simply $R\mathrm{Hom}(F', F'')$.

Obstructions in deformation theory: X smooth alg var$/\mathbb{C}$

canonical obstruction class map

$$\rho : \{1^{st}\text{-order deformations of } X\} \longrightarrow H^2(X, T_X)$$

A 1^{st}-order deformation can be extended to a second order deformation iff its obstruction class vanishes.

Q: While elements in $H^0(X, T_X)$ and $H^1(X, T_X)$ have geometric meanings in deformation theory, what is the meaning of $H^2(X, T_X)$ and $H^n(X, T_X)$ $n \geq 2$?

Prop: Let R be the square-zero extension $\mathbb{C} \oplus \mathbb{C}[n]$

(a) $\forall$ deformation X' of X over R

$$\{\text{Autom of } X' \text{ that are id on } X_0\}/_{\text{homotopy}} \simeq H^n(X, T_X)$$

(b) $\{\text{Deformations of } X \text{ over } R\}/_{\text{homotopy}} \simeq H^{n+1}(X, T_X)$

Relation with the obstruction class map:

We have a pullback diagram of cdgas:

$$\begin{tikzcd} \mathbb{C}[\varepsilon]/(\varepsilon^3) \arrow[r] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{C}[\varepsilon]/(\varepsilon^2) \arrow[r] & \mathbb{C} \oplus \mathbb{C}[1] \end{tikzcd}$$

$\rightsquigarrow$ pushout diagram of dg schemes:

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^3) & \operatorname{Spec} \mathbb{C} \arrow[l] \\
\operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2) \arrow[u] & \operatorname{Spec} \mathbb{C}\oplus\mathbb{C}[1] \arrow[l] \arrow[u]
\end{tikzcd}$$

Therefore, a 1^{st}-order deformation $X_1/_{\operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2)}$ of X extends to a 2^{nd}-order deformation over $\mathbb{C}[\varepsilon]/(\varepsilon^3)$ iff the pullback of X_1 to $\operatorname{Spec} \mathbb{C}\oplus\mathbb{C}[1]$ is a trivial deformation of X over $\operatorname{Spec} \mathbb{C}\oplus\mathbb{C}[1]$.
This pullback is equivalent to the obstruction class map.

II. Infinity categories

In derived geometry, we study objects up to homotopy equivalence. In order to keep track of all homotopies, we use the language of infinity categories. We use simplicial sets as models for homotopy types.

1. Simplicial sets

Def: $[n] := \{0 < 1 < \cdots < n\}$ finitely linearly ordered set

The category of combinatorial simplices Δ:

Obj: $[n]$, $n \geq 0$

Mor: $\alpha : [m] \longrightarrow [n]$ non-decreasing

cat of sets

A simplicial set is a functor $S_\bullet : \Delta^{op} \longrightarrow \mathcal{S}et$

$S_n := S_\bullet([n])$ the set of n-simplices of $S_\bullet$. S_0: vertices S_1: edges

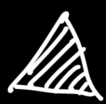
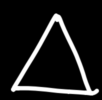
$S_\bullet(\alpha : [m] \longrightarrow [n]) : S_n \longrightarrow S_m$ describes how the simplices are glued together.

$\mathcal{S}et_\Delta := \operatorname{Fun}(\Delta^{op}, \mathcal{S}et)$ the category of simplicial sets.

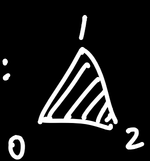
$\Delta^n := \mathrm{Hom}_\Delta(-,[n]) \in \mathrm{Set}_\Delta$ standard n-simplex

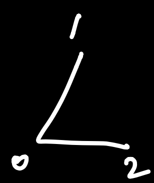
$\forall$ simplicial set $S_\bullet$, Yoneda lemma $\Rightarrow S_n \simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, S_\bullet)$

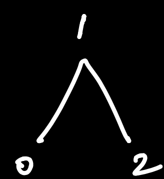
$\partial\Delta^n :=$ the simplicial subset of Δ^n whose m-simplices are non-decreasing
boundary non-surjective maps $\alpha : [m] \to [n]$.

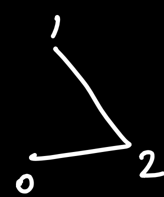
Δ^2:  $\partial\Delta^2$: 

For $0<i<n$, the i^{th} horn Λ^n_i of Δ^n is the simplicial subset of $\partial\Delta^n$ whose m-simplices are non-decreasing maps $\alpha : [m] \to [n]$ s.t. $\alpha([m]) \cup \{i\} \neq [n]$.

Δ^2: 

Λ^2_0: 

Λ^2_1: 

Λ^2_2: 

Def: A simplicial set $S_\bullet$ is a Kan complex if $\forall\, 0 \le i \le n$, $\forall$ map $\sigma_0 : \Lambda^n_i \to S_\bullet$ can be extended to an n-simplex $\sigma : \Delta^n \to S_\bullet$

$\mathcal{K}an$: the category of Kan complexes

$S_\bullet, T_\bullet$ simplicial sets, $f, g : S_\bullet \to T_\bullet$. A simplicial homotopy from f to g is map of simplicial sets $h : S_\bullet \times \Delta^1 \to T_\bullet$ s.t. $h|_{S_\bullet \times \{0\}} = f$ and $h|_{S_\bullet \times \{1\}} = g$. In this case, f and g are said to be simplicially homotopic.

If $T_\bullet$ is a Kan complex, this is an equivalence relation.

The homotopy category of Kan complexes $h\mathcal{K}an$: Obj: Kan complexes
Mor: simplicial homotopy classes of maps.

2. ∞-categories

Q: How to incorporate homotopy data into a ordinary category?

Recall: $\mathcal{C}$ category $\rightsquigarrow$ simplicial set $N(\mathcal{C})$, called nerve

$$N(\mathcal{C})_n := \mathrm{Fun}(\underbrace{\bullet \to \bullet \to \cdots \to \bullet}_{n+1}, \mathcal{C}) = \{X_0 \to \cdots \to X_n, X_i \in \mathcal{C}\}$$

Face maps given by compositions, degeneracy maps given by insertions of identities.

Prop: The construction $\mathcal{C} \mapsto N(\mathcal{C})$ determines a fully faithful embedding from the category $\mathcal{C}at$ of small categories to the category $\mathcal{S}et_\Delta$ of simplicial sets. The essential image of this embedding consists of those simplicial sets $S_\bullet$ with the following property:

$$\forall\, 0<i<n, \quad \forall\ \begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & S_\bullet \\ \Delta^n \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

Def: An ∞-category $\mathcal{C}$ (modeled as a weak Kan complex) is a simplicial set s.t. $\forall\, 0<i<n$,

$$\forall\ \begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

uniqueness dropped

Example: $i=1$, $n=2$

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta \arrow[ur, dashed] & \end{tikzcd}$$

"existence of composition, but non-unique"

Basic notions:

$\mathcal{C}$ ∞-category. Its 0-simplices are called objects, 1-simplices morphisms

$\forall$ objects $X, Y \in \mathcal{C}$, mapping space $\mathrm{Map}_{\mathcal{C}}(X,Y)$ is the Kan complex whose n-simplices are maps $\Delta^n \times \Delta^1$ to $\mathcal{C}$ which sends $\Delta^n \times \{0\}$ to the vertex X and $\Delta^n \times \{1\}$ to the vertex Y.

$\mathcal{C} \rightsquigarrow$ homotopy category $h\mathcal{C} := \begin{cases} \text{Obj: same as } \mathcal{C} \\ \text{Mor: } \pi_0 \mathrm{Map}_{\mathcal{C}}(X,Y) \end{cases}$

$\mathcal{H}$-enriched homotopy category $h\mathcal{C} := \begin{cases} \text{Obj: same as } \mathcal{C} \\ \text{Mor: } [\mathrm{Map}_{\mathcal{C}}(X,Y)] \in \mathcal{H} \end{cases}$

$\uparrow$ homotopy category of spaces

A morphism in $\mathcal{C}$ is called an equivalence if its image in $h\mathcal{C}$ is an isomorphism. Two objects in $\mathcal{C}$ are equivalent if $\exists$ an equivalence between them.

$\mathcal{C}, \mathcal{D}$ ∞-categories. A functor F from $\mathcal{C}$ to $\mathcal{D}$ is a map of simplicial sets. The ∞-category of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) := \mathrm{Map}_{\mathrm{Set}_\Delta}(\mathcal{C}, \mathcal{D})$, its set of n-simplices are by definition $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathcal{C} \times \Delta^n, \mathcal{D})$

$F: \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if $\exists$ functor $G: \mathcal{D} \to \mathcal{C}$ s.t. $G \circ F$ is equivalent to $\mathrm{id}_{\mathcal{C}}$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ and $F \circ G$ is equivalent to $\mathrm{id}_{\mathcal{D}}$ in $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$.

$F: \mathcal{C} \to \mathcal{D}$ is essentially surjective if $hF: h\mathcal{C} \to h\mathcal{D}$ is essentially surj.

$F: \mathcal{C} \to \mathcal{D}$ is fully faithful if $hF: h\mathcal{C} \to h\mathcal{D}$ is fully faithful on the $\mathcal{H}$-enriched homotopy categories, i.e. $\forall X, Y \in \mathcal{C}$,

$$\mathrm{Map}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(F(X), F(Y)) \text{ is a homotopy equiv.}$$

Equivalence $\iff$ fully faithful + essentially surj.

$\mathcal{C}$ ∞-category. $(h\mathcal{C})' \subset h\mathcal{C}$ subcat. We form the pullback diagram of simplicial sets

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ N((h\mathcal{C})') \arrow[r] & N(h\mathcal{C}) \end{tikzcd}$$

$\mathcal{C}'$ is called the subcat of $\mathcal{C}$ spanned by $(h\mathcal{C})'$

$\mathcal{C}' \subset \mathcal{C}$ is called a full subcategory if $(h\mathcal{C})' \subset h\mathcal{C}$ is a full subcat.

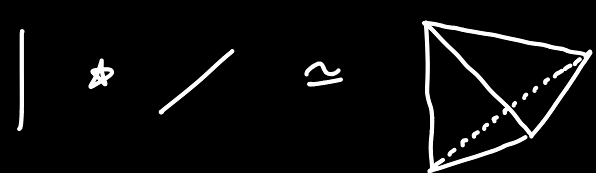
$\mathcal{C}$ ∞-cat. An obj $X \in \mathcal{C}$ is final if $\forall\, Y \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(Y, X)$ is contractible.

initial $\qquad \mathrm{Map}_{\mathcal{C}}(X, Y)$

3. Limits and colimits

Def: S, S' simplicial sets $\rightsquigarrow$ join $S \star S'$: $(S \star S')_n := S_n \cup S'_n \cup \bigcup_{i+j=n-1} S_i \times S'_j$

Ex: $\Delta^i \star \Delta^j \simeq \Delta^{i+j+1}$



Prop (Joyal): The join of two weak Kan complexes is a weak Kan complex.

Def: K simplicial set. Left cone $K^{\triangleleft} := \Delta^0 \star K$ Right cone $K^{\triangleright} := K \star \Delta^0$.

Prop (Joyal): $p: K \to S$ map of simplicial sets. Then $\exists$ a simplicial set $S_{/p}$ with the following universal property:

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(Y, S_{/p}) = \mathrm{Hom}_p(Y \star K, S)$$

subset consisting of f s.t. $f|_K = p$

Proof: $(S_{/p})_n := \mathrm{Hom}_p(\Delta^n \star K, S)$

Prop: $p: K \to \mathcal{C}$ map of simplicial sets, $\mathcal{C}$ is weak Kan. Then $\mathcal{C}_{/p}$ is weak Kan.

called overcategory

If $p: K = \Delta^0 \mapsto X \in \mathcal{C}$, we denote $\mathcal{C}_{/X} := \mathcal{C}_{/p}$

Dually, replacing $Y \star K$ by $K \star Y$ $\rightsquigarrow$ undercategory $\mathcal{C}_{p/}$ $\mathcal{C}_{X/}$

Def: $\mathcal{C}$ ∞-cat, $p: K \to \mathcal{C}$ map of simplicial sets.

A colimit for p is an initial obj of $\mathcal{C}_{p/}$

limit final $\mathcal{C}_{/p}$.

A colimit diagram is the associated $\bar{p}: K^{\triangleright} \to \mathcal{C}$ extending p.

We will simply refer to $\bar{p}(\infty) \in \mathcal{C}$ as the colimit of p.

distinguished vertex of $\triangleright$

Same for limit.