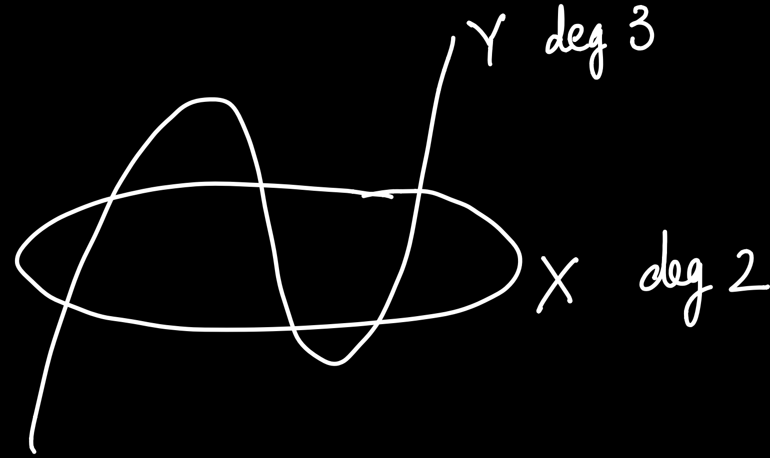
Introduction to Derived Geometry

I. Motivations

1. Bézout theorem

$X, Y \subset \mathbb{C}P^2$ smooth algebraic curves

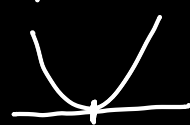
X: deg a, Y: deg b



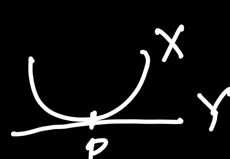
Bézout theorem: If X and Y meet transversely, then the intersection $X \cap Y$ has ab points.

$$\# X \cap Y = \deg X \cdot \deg Y$$

Q: What about non-transverse intersections?

- proper non-transverse intersection  $\dim X \cap Y = 0$.
- non-proper intersection  self-intersection

If we want Bézout theorem to continue to hold for non-transverse intersections, we need to reinterpret $X \cap Y$, ie. equip $X \cap Y$ with more structures than just a set.

Proper non-transverse intersection: 

We equip each intersection point p with a multiplicity:

$$\mathrm{mult}(p) := \dim \mathcal{O}_{X,p} \underset{\mathcal{O}_{\mathbb{CP}^2,p}}{\otimes} \mathcal{O}_{Y,p}$$

then $\displaystyle\sum_{p \in X \cap Y} \mathrm{mult}(p) = \deg X \cdot \deg Y$

Warning from comm alg: $\otimes$ is not exact, we have Tor functors.

Indeed, the above multiplicity formula is wrong in higher dim for singular subvarieties, and should be corrected by Tor.

Assume $X, Y \subset \mathbb{CP}^n$ (singular) subvar, $\dim X + \dim Y = n$

$X \cap Y$ properly i.e. $\dim X \cap Y = 0$.

For every $p \in X \cap Y$, the correct multiplicity should be

$$\mathrm{mult}(p) := \sum (-1)^i \dim \mathrm{Tor}_i^{\mathcal{O}_{\mathbb{CP}^n,p}}(\mathcal{O}_{X,p}, \mathcal{O}_{Y,p})$$

Serre's intersection formula

Note that $\mathrm{Tor}_0 = \otimes$ product, Tor_i $i>0$ are correction terms

Then Bezout thm still holds: $\displaystyle\sum_{p \in X \cap Y} \mathrm{mult}(p) = \deg X \cdot \deg Y$.

Q: What about non-proper intersections? $\dim X \cap Y > 0$.

2. Review of Tor functors

R comm ring, A, B R-modules. $\mathrm{Tor}_i^R(A, B)$

Choose a proj resolution of A:

$$\cdots\cdots \to P_2 \to P_1 \to P_0 \to A \to 0 \qquad P_i : \text{proj } R\text{-modules}$$

then $\mathrm{Tor}_i^R(A,B)$ are the homology groups of the chain complex

$$\cdots \to P_2 \otimes_R B \to P_1 \otimes_R B \to P_0 \otimes_R B \quad =: \quad A \overset{L}{\underset{R}{\otimes}} B$$

derived tensor product

Recall Serre's intersection formula for proper intersections:

$$\mathrm{mult}(p) := \sum (-1)^i \dim \mathrm{Tor}_i^{\mathcal{O}_{\mathbb{CP}^n,p}}(\mathcal{O}_{X,p}, \mathcal{O}_{Y,p})$$

$$\overset{\text{Rewrite}}{=} \chi\left(\mathcal{O}_{X,p} \overset{L}{\underset{\mathcal{O}_{\mathbb{CP}^n,p}}{\otimes}} \mathcal{O}_{Y,p}\right)$$

We can rewrite Bezout theorem $\displaystyle\sum_{p \in X \cap Y} \mathrm{mult}(p) = \deg X \cdot \deg Y$

as $$\chi\left(\mathcal{O}_X \overset{L}{\underset{\mathcal{O}_{\mathbb{CP}^d}}{\otimes}} \mathcal{O}_Y\right) = \deg X \cdot \deg Y$$

This reformulation actually generalizes to arbitrary non-proper non-transverse intersections.

Example: $C \subset \mathbb{CP}^2$ deg d. Self-intersection $C \cap C$

Let's compute $\mathcal{O}_C \overset{L}{\underset{\mathcal{O}_{\mathbb{CP}^2}}{\otimes}} \mathcal{O}_C$

$$0 \to \mathcal{O}(-C) \to \mathcal{O} \to \mathcal{O}_C \to 0$$

$$\otimes \mathcal{O}_C \quad \rightsquigarrow \quad \mathcal{O}(-C) \otimes \mathcal{O}_C \to \mathcal{O}_C$$

$$\chi\left(\mathcal{O}_C \overset{L}{\underset{\mathcal{O}_{\mathbb{P}^2}}{\otimes}} \mathcal{O}_C\right) = \chi\left(\mathcal{O}(-C)\otimes\mathcal{O}_C \to \mathcal{O}_C\right)$$

$$= \chi(\mathcal{O}_C) - \chi(\mathcal{O}(-C)\otimes\mathcal{O}_C)$$

Riemann-Roch

$$= (1-g) - (-d^2 - g + 1) = d^2$$

Moral: For Bezout theorem to hold in full generality, we would like to equip the set-theoretic intersection $X \cap Y$ with the derived tensor product $\mathcal{O}_X \overset{L}{\underset{\mathcal{O}_{\mathbb{C}P^d}}{\otimes}} \mathcal{O}_Y$.

Remark: Recall $P_\bullet \to A$ projective resolution

When A, B are R-algebras, we can also enhance $P_\bullet$

$\rightsquigarrow$ a commutative differential graded R-algebra cdga

(i) multiplications $P_m \underset{R}{\otimes} P_n \to P_{m+n} \rightsquigarrow \bigoplus_{n\geq 0} P_n$ comm graded ring

$xy = (-1)^{|x||y|} yx$

(ii) differential $d: P_* \to P_{*-1}$ satisfies the graded leibniz rule:

$$d(xy) = (dx)y + (-1)^{|x|} x\,dy$$

Then $P_\bullet \underset{R}{\otimes} B =: A \overset{L}{\underset{R}{\otimes}} B$ inherits a cdga structure.

Def: A dg scheme $/\mathbb{C}$ $(X, \mathcal{O}_X)$ consits of a topological space X and a sheaf of cdga$/\mathbb{C}$ $\mathcal{O}_X$ on X s.t.

(a) $(X, H_0(\mathcal{O}_X))$ is a scheme called the truncation of X

(b) $H_n(\mathcal{O}_X)$ is a quasi-coherent sheaf on

(c) $H_n(\mathcal{O}_X)$ vanish for $n<0$.

Rem: 1) Language of ∞-categories

2) cdga $\rightsquigarrow$ simplicial rings derived scheme

3) multiplications on are strictly comm and assoc.

Not natural or convenient. $\rightsquigarrow$ relax $\rightsquigarrow$ E_∞-rings
spectral scheme

cdga, simplicial rings, E_∞-rings are all equiv. over char 0.

3. Enumerative geometry

Problem: Given X sm proj var $/\mathbb{C}$, $\beta \in H_2(X,\mathbb{Z})$

want to count algebraic curves in X with class β and some constraints

Example: • Count rational curves in $\mathbb{P}^2$ of deg d passing through $3d-1$ general pts.

Answer: 1 1 12 620 87304 26312976 14616808192 ....

• Count degree-d rational curves on a general quintic $X \subset \mathbb{CP}^4$.

Answer: 2875 609250 317206375 242467530000 ....

Idea: let $\mathcal{M}$ be the moduli space of curves in X with class β and constraints.

We want to count the number of pts in $\mathcal{M}$.

Similar to Bezout thm: multiplicities, $\dim \mathcal{M} > 0$.

We cannot naïvely take the cardinality of $\mathcal{M}$.

We should endow $\mathcal{M}$ with an extra structure: derived structure

Historically: perfect obstruction theory Li-Tian Behrend-Fantechi

4. Homotopy theory

Cdga and $\mathbb{E}_\infty$-rings occur naturally in homotopy theory:

de Rham complex: M smooth manifold, $H^*(M,\mathbb{R})$ can be computed by the de Rham complex: $\Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \Omega^2_M \xrightarrow{d} \cdots$ cdga

More generally, $\forall$ topological space X

$\xrightarrow{\text{Sullivan}}$ polynomial de Rham complex $C^*_{dR}(X;\mathbb{Q})$ cdga

Theorem (Sullivan): X simply conn top space s.t. $\dim H^n(X;\mathbb{Q}) < +\infty$. Then the rational homotopy type of X can be recovered from its polynomial de Rham complex $C^*_{dR}(X;\mathbb{Q})$.

More precisely, the canonical map $X \longrightarrow X_{\mathbb{Q}} := \mathrm{Map}_{cdga}(C^*_{dR}(X;\mathbb{Q}), \mathbb{Q})$ is a rational homotopy equivalence.

Reformulate: $\hat{X} := \mathrm{Spec}\, C^*_{dR}(X;\mathbb{Q})$ dg-scheme "schematization of X"

Then $X_{\mathbb{Q}} = \hat{X}(\mathbb{Q})$ the space of $\mathbb{Q}$-valued points of $\hat{X}$ is rationally homotopy equivalent to X.

More generally, $\forall$ top space X, $\forall$ field k, the singular chain complex $C^*(X;k)$ has the structure of an $\mathbb{E}_\infty$-algebra $/k$.

Thm (Mandell): X simply conn top space, $\dim H^n(X, \mathbb{F}_p) < +\infty$.

Then the canonical map $X \longrightarrow X_p^\wedge := \mathrm{Map}_{E_\infty}(C^*(X;\mathbb{F}_p), \overline{\mathbb{F}}_p)$ is an isom on $\mathbb{F}_p$-cohomology. $\pi_n X_p^\wedge \simeq$ p-adic completion of $\pi_n X$.

5. Derived categories.

Fourier–Mukai transform: E elliptic curve$/\mathbb{C}$

$$\begin{tikzcd} & E \times E \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\ E & & E \end{tikzcd}$$

$\mathcal{P}$: line bundle on $E \times E$ corresponding to the Cartier divisor $\Delta - \{e\} \times E - E \times \{e\}$.

Consider functor
$$QCoh(E) \longrightarrow QCoh(E)$$
$$\mathcal{F} \longmapsto \pi_{1*}(\mathcal{P} \otimes \pi_0^* \mathcal{F})$$

Poorly behaved: neither exact, nor faithfull.

$\rightsquigarrow$ Great improvement by passing to derived categories:

FM transform: $DQCoh(E) \longrightarrow DQCoh(E)$ is an equivalence
$$\mathcal{F} \longmapsto R\pi_{1*}(\mathcal{P} \otimes \pi_0^* \mathcal{F})$$

Theorem (Bondal–Orlov): X sm proj var$/$field k. K_X is either ample or antiample.

then X is determined by $D^b Coh(X) \subset DQCoh(E)$

chain complexes of bounded coherent cohomology

Base change theorem