Doubling integrals for Brylinski-Deligne extensions of classical groups

蔡園青 (Yuanqing Cai)
京都大学

2020 年 8 月
燕京云数論 (POINTS)

Automorphic L-functions, I

- F: number field with adele ring $\mathbb{A}$
- G: split reductive group over F
- π: irreducible cuspidal automorphic representation of $\mathrm{G}(\mathbb{A})$
- $\pi = \otimes'_v \pi_v$: restricted tensor product, where π_v is unramified for almost all places
- $\mathrm{G}^\vee$: dual group of G
- $\rho : \mathrm{G}^\vee \to \mathrm{GL}_n(\mathbb{C})$.

Automorphic L-functions, II

For an unramified place v:

- Satake isomorphism: π_v unramified $\leftrightarrow$ Satake parameter t_v (a semi-simple conjugacy class in $\mathrm{G}^\vee$).
- $q_v = \#\mathcal{O}_v/\mathcal{P}_v$
- Local L-function:

$$L_v(s, \pi_v, \rho) = \frac{1}{\det(I - \rho(t_v)q_v^{-s})}.$$

Fix a finite set of places such that π_v is unramified if $v \notin S$. Define global partial L-function:

$$L^S(s, \pi, \rho) = \prod_{v \notin S} L_v(s, \pi_v, \rho).$$

Automorphic L-functions, III

Basic question

Show that $L^S(s, \pi, \rho)$ admits meromorphic continuation to $\mathbb{C}$, and has a functional equation for $s \mapsto 1 - s$.

Basic method

Find a global integral that represents the desired L-function.

To obtain an Euler product, almost all examples use some kind of multiplicity one results.

Examples

- Rankin-Selberg integrals for $\mathrm{GL}_m \times \mathrm{GL}_n$ (Jacquet – Piatetski-Shapiro – Shalika): uniqueness of Whittaker models
- Langlands-Shahidi method: uniqueness of Whittaker models
- Godement-Jacquet integrals: matrix coefficients (works for all cuspidal representations)
- Doubling integrals: matrix coefficients

Covering groups

Notable example (metaplectic groups)

$$1 \to \mu_2 \to \mathrm{Mp}_{2n} \to \mathrm{Sp}_{2n} \to 1.$$

A covering group is typically of the form (central extension)

$$\text{locally: } 1 \to \mu_n \to \overline{G(F_v)} \to G(F_v) \to 1$$
$$\text{globally: } 1 \to \mu_n \to \overline{G(\mathbb{A})} \to G(\mathbb{A}) \to 1$$

More general construction

Explicit 2-cocycle in $\mathrm{H}^2(G, \mu_n)$: Weil, Kubota, Moore, Matsumoto, Kazhdan-Patterson, Banks-Levi-Sepanski...

Recently

A class of covering groups following Brylinski-Deligne (2001), which is more algebraic and functorial.

Brylinski-Deligne extensions, I

- G: split reductive group over F
- T: maximal split torus of G
- Y: cocharacter lattice of T

$\mathbf{CExt}(\mathrm{G}, \mathrm{K}_2)$

the (Picard) category of central extensions

$$1 \to \mathrm{K}_2 \to \overline{\mathrm{G}} \to \mathrm{G} \to 1$$

as sheaves of groups on the big Zariski site over $\mathrm{Spec}(F)$.

Brylinski-Deligne extensions, II

Classification by Brylinski-Deligne

$\mathbf{CExt}(\mathrm{G}, \mathrm{K}_2)$ is equivalent to $\mathrm{BD}(\mathrm{G}, \mathrm{T})$ which is a simpler and concrete category.

When G is simply-connected, then $\overline{\mathrm{G}}$ is classified by a Weyl invariant quadratic form Q on Y.

Example

$\mathrm{G} = \mathrm{Sp}_{2r}$. The Dynkin diagram is given by

$$\alpha_1 = \alpha_2 - \alpha_3 - \cdots - \alpha_r \ .$$

A Weyl invariant quadratic form is determined by $Q(\alpha_1^\vee)$.

The metaplectic groups in the usual sense (double cover of symplectic groups) can be obtained from this K_2-extension.

Brylinski-Deligne extensions, III

Connection with topological covering groups

(Matsumoto) For a field L,

$$\mathrm{K}_2(L) = L^\times \otimes_{\mathbb{Z}} L^\times / \langle a \otimes (1-a) | a \neq 0, 1 \rangle.$$

For a local field L, the Hilbert symbol

$$(-,-) : L^\times \times L^\times \to \mathrm{K}_2(L) \to \mu(L)$$

where $\mu(L)$ is the group of roots of unity.

Topological covering groups, I

Let v be a local place of F with $\mu_n \subset F_v^\times$, get

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{K}_2(F_v) \arrow[r] \arrow[d, "(-,-)_n"] & \overline{\mathrm{G}}(F_v) \arrow[r] \arrow[d] & \mathrm{G}(F_v) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mu_n \arrow[r] & \overline{G}_v \arrow[r] & G_v = \mathrm{G}(F_v) \arrow[r] & 1
\end{tikzcd}$$

Globally, get

$$1 \to \mu_n \to \overline{G}(\mathbb{A}) \to G(\mathbb{A}) \to 1.$$

Topological covering groups, II

The K_2-extension carries information that allows us to do research in automorphic forms (without the use of cocycles):

- a canonical splitting over any unipotent subgroup of G
- a natural splitting over $\mathrm{G}(\mathcal{O}_v)$ at almost all v. (Satake isomorphism)
- a natural splitting $G(F) \to \overline{G}(\mathbb{A})$. This allows us to talk about automorphic forms.

Automorphic representations on covering groups, I

Recent progress

- Gan-Gao-Weissman 2018: L-groups and the Langlands program for covering groups

Most of the basic theory of automorphic representations holds in this case.
For example, fix a finite set of places of S which is large enough. Given a representation $\rho : \overline{\mathrm{G}}^{\vee} \to \mathrm{GL}_m(\mathbb{C})$, one can define the (partial) automorphic L-function

$$L^S(s, \pi, \rho) = \prod_{v \notin S} L(s, \pi_v, \rho).$$

Automorphic representations on covering groups, II

Question

- Meromorphic continuation and functional equations for some L-functions for covering groups?

Obstruction

"Multiplicity one results" fail in general for covering groups.
In particular, uniqueness of Whittaker models fails in general for covering groups.

This means that it is much harder to obtain Euler product for covering groups.

Recent progress, I

Langlands-Shahidi L-functions for covering groups

Gao 2018: calculate constant terms of Eisenstein series on BD covering groups.

Consequence: meromorphic continuation of many interesting L-functions.

However, since uniqueness of Whittaker models fails, it seems difficult to deduce functional equations.

Recent progress, II

Doubling integrals

- Piatetski-Shapiro – Rallis: standard L-functions for classical groups
- C.-Friedberg-Ginzburg-Kaplan: tensor product L-function of a classical group and a general linear group, generalizing the doubling integrals. (Twisted doubling integrals.)
- Kaplan 2019: covers of symplectic groups.
- C. 2019: set up the global integrals for Brylinski-Deligne extensions of all classical groups.
  (Excluding certain covers of unitary groups due to an extra complication.)

The doubling zeta integrals, I

- $\mathcal{W} = (W, \langle\ ,\ \rangle)$: a quadratic space over F
- W: vector space over F of dimension m
- $\langle\ ,\ \rangle$: non-degenerate bilinear form on W
- $G = G(\mathcal{W})$: isometry group of $\mathcal{W}$.

Examples: $\mathrm{Sp}_m, \mathrm{O}_m$.

Remark

One can also include cases such as: SO_m, inner forms of $\mathrm{Sp}_m, \mathrm{SO}_m$ and unitary groups.

The doubling zeta integrals, II

The doubling map

Define $\mathcal{W}^\square = (W^\square, \langle\ ,\ \rangle^\square)$ where

$$W^\square = W_+ \oplus W_-$$

and

$$\langle (x_+, x_-), (y_+, y_-) \rangle^\square = \langle x_+, y_+ \rangle - \langle x_-, y_- \rangle.$$

Define $G^\square = G(\mathcal{W}^\square)$.
The group $G \times G$ acts on $W^\square$ via

$$(g_1, g_2) \cdot (x_+, x_-) = (g_1 x_+, g_2 x_-).$$

This gives a homomorphism

$$\iota : G \times G \to G^\square.$$

The doubling zeta integrals, III

Siegel parabolic subgroup

Define

$$W^{\Delta} = \{(x, x) \in W^{\square} : x \in W\}.$$

Then $\langle\ ,\ \rangle^{\square}|_{W^{\Delta} \times W^{\Delta}} = 0$, i.e. W^{Δ} is a totally isotropic subspace. This gives a Siegel parabolic subgroup $P(W^{\Delta}) = M(W^{\Delta})N(W^{\Delta})$ or $P = MN$.

Eisenstein series

- $\chi : F^{\times} \backslash \mathbb{A}^{\times} \to \mathbb{C}^{\times}$
- $\chi \circ \det$ is a character of $\mathrm{GL}_m(F) \backslash \mathrm{GL}_m(\mathbb{A})$; this gives a character of $M(F) \backslash M(\mathbb{A})$
- $I(s, \chi) = \mathrm{Ind}_{P(\mathbb{A})}^{G^{\square}(\mathbb{A})} (\chi \circ \det) \cdot \delta_P^s$
- $f^{(s)} \in I(s, \chi)$, one attaches an Eisenstein series

$$E(f^{(s)})(g) = \sum_{\gamma \in P(F) \backslash G^{\square}(F)} f^{(s)}(\gamma g).$$

The doubling integrals, IV

- π: irreducible cuspidal representation of $G(\mathbb{A})$
- $\xi_1 \in \pi$ and $\xi_2 \in \pi^\vee$

Global zeta integral

$$Z(\xi_1 \boxtimes \xi_2, f^{(s)}) =$$

$$\int_{G(F)\backslash G(\mathbb{A}) \times G(F)\backslash G(\mathbb{A})} \xi_1(g_1)\xi_2(g_2)E(f^{(s)})(\iota(g_1, g_2))\ dg_1\ dg_2.$$

The doubling integrals, V

Unfolding

$$Z(\xi_1 \boxtimes \xi_2, f^{(s)}) = \int\limits_{G(\mathbb{A})} \mathcal{P}(\pi(g)\xi_1 \boxtimes \xi_2) f^{(s)}(\iota(g, e))\ dg,$$

where

$$\mathcal{P}(\xi_1 \boxtimes \xi_2) = \int\limits_{G(F)\backslash G(\mathbb{A})} \xi_1(g)\xi_2(g)\ dg.$$

This is an Euler product since $\mathcal{P}$ is decomposable and local components of one-dimensional representations are one-dimensional.

Unramified calculation

$$Z(\xi_1 \boxtimes \xi_2, f^{(s)}) \approx L(s, \pi \times \chi).$$

The twisted doubling integrals, I

Goal: tensor product L-function $\mathrm{G} \times \mathrm{GL}_k$.

The doubling map

- $\mathcal{W}^{\square,k} = (W^{\square,k}, \langle\ ,\ \rangle^{\square,k})$ where
- $W^{\square,k} = W_1^{\square} \oplus \cdots \oplus W_k^{\square}$
- $\langle\ ,\ \rangle^{\square,k} = \langle\ ,\ \rangle_1^{\square} \oplus \cdots \oplus \langle\ ,\ \rangle_k^{\square}$.
- $\mathrm{G}^{\square,k} = G(\mathcal{W}^{\square,k})$
- $(g_1, g_2) \in \mathrm{G} \times \mathrm{G}$ acts on $W^{\square,k}$ via

$$\begin{aligned}(g_1,g_2)\cdot&(x_{1+},x_{1-},x_{2+},x_{2-},\cdots,x_{k+},x_{k-})\\ &=(g_1x_{1+},\quad \mathbf{g_2x_{1-}},\quad g_1x_{2+},g_1x_{2-},\cdots,g_1x_{k+},g_1x_{k-}).\end{aligned}$$

This defines a homomorphism $\iota_k : \mathrm{G} \times \mathrm{G} \to \mathrm{G}^{\square,k}$.

The twisted doubling integrals, II

Siegel parabolic subgroup

Define

$$W^{\Delta,k} = W_1^{\Delta} \oplus \cdots \oplus W_k^{\Delta}.$$

This is a maximal totally isotropic subspace of $W^{\square,k}$.
Define $P = P(W^{\Delta,k}) \subset \mathrm{G}^{\square,k}$

Fourier coefficient associated with the nilpotent orbit $((2k-1)^m 1^m)$

In one moment.

Degenerate Whittaker coefficients, I

Nilpotent orbits

Let $\mathcal{N}_{\mathfrak{g}}$ be the set of nilpotent elements in a semisimple Lie algebra $\mathfrak{g}$. Under the adjoint action, it becomes a disjoint union of nilpotent orbits.

- GL_m case: the theory of Jordan canonical form
- Nilpotent orbits of GL_m are in bijection with partitions of m.
- For classical groups, nilpotent orbits are in bijection with partitions with additional assumptions.
- There is a partial order on the set of nilpotent orbits

Degenerate Whittaker models/coefficients, II

Examples

The orbit (3^2):

$$\begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & & 0 & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 \end{pmatrix} \text{ or } f_{(3^2)} = \begin{pmatrix} 0 & 0 & & & & \\ 0 & 0 & & & & \\ 1 & 0 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & & \\ & & 1 & 0 & 0 & 0 \\ & & 0 & 1 & 0 & 0 \end{pmatrix}$$

Observe: image of

$$\mathrm{GL}_2 \to \mathrm{GL}_6, \qquad g \mapsto \mathrm{diag}(g, g, g)$$

lies in the stabilizer of $f_{(3^2)}$.

Generalization: for the orbit (k^n), its stabilizer contains the image of

$$\mathrm{GL}_n \to \mathrm{GL}_{kn}, \qquad g \mapsto \mathrm{diag}(g, g, \cdots, g).$$

Degenerate Whittaker models/coefficients, III

Example: orbit $(3^2 1^2)$

$$\begin{pmatrix} 0 & & & & & & & \\ 1 & 0 & & & & & & \\ & 1 & 0 & & & & & \\ & & & 0 & & & & \\ & & & 1 & 0 & & & \\ & & & & 1 & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix} \text{ or } f_{(3^2 1^2)} = \begin{pmatrix} 0 & 0 & & & & & & \\ 0 & 0 & & & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & 0 & & & & \\ & & 1 & 0 & 0 & 0 & & \\ & & 0 & 1 & 0 & 0 & & \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \end{pmatrix}$$

Note: the image

$$\mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GL}_8, \qquad (g_1, g_2) \mapsto \mathrm{diag}(g_1, g_1, g_1, g_2)$$

lies in the stabilizer of $f_{(3^2 1^2)}$.

Degenerate Whittaker models/coefficients, IV

This can be generalized to the orbit $((2k-1)^n 1^n)$:
one can choose a nice representative so that the stabilizer of this representative contains the image of

$$\mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_{2kn}, \qquad (g_1, g_2) \mapsto \mathrm{diag}(g_1, g_2, g_1, g_1, \cdots, g_1, g_1).$$

Degenerate Whittaker models/coefficients, V

The Whittaker model

This is attached to the orbit (n):
For example, if $G = GL_n$ and

$$N_n = \left\{ u = \begin{pmatrix} 1 & u_{12} & * & \cdots & * \\ & 1 & u_{23} & \cdots & * \\ & & 1 & \cdots & * \\ & & & & \vdots \\ & & & & 1 \end{pmatrix}. \right\}$$

A generic character is of the form

$$\psi_{N_n}(u) = \psi(u_{12} + u_{23} + \cdots + u_{n-1,n})$$

where ψ is a nontrivial additive character of $F\backslash\mathbb{A}$.

Degenerate Whittaker models/coefficients, VI

One can write this as

$$\psi_{N_n}(u) = \psi(\mathrm{tr}(f_{(n)} u))$$

where

$$f_{(n)} = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & & & \vdots \\ & & & 1 & 0 \end{pmatrix}$$

This defines a Fourier coefficient associated to the nilpotent orbit (n).

Another example

Consider

$$N_{(3^2)} = \left\{ u = \begin{pmatrix} I_2 & X_1 & Y \\ & I_2 & X_2 \\ & & I_2 \end{pmatrix} \in \mathrm{GL}_6 \right\}$$

and

$$\psi_{(3^2)}(u) = \psi(\mathrm{tr}(X_1 + X_2)).$$

Equivalently,

$$\psi_{(3^2)}(u) = \psi(\mathrm{tr}(f_{(3^2)} u)).$$

Recall that $f_{(3^2)} = \begin{pmatrix} 0 & & \\ I_2 & 0 & \\ & I_2 & 0 \end{pmatrix}$. This defines a Fourier coefficient associated to the nilpotent orbit (3^2).

Degenerate Whittaker models/coefficients, VII

Whittaker pair (S, φ)

- $(S, \varphi) \in \mathfrak{g} \times \mathfrak{g}^*$ such that S is rational semi-simple and $\mathrm{ad}_*(S)(\varphi) = -2\varphi$.
- Using the Killing form, $\varphi \leftrightarrow f \in \mathfrak{g}$, and f is nilpotent

Degenerate Whittaker model

- $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ according to eigenvalues of S; assume that 1 is not an eigenvalue
- $\mathfrak{n} = \oplus_{i>1} \mathfrak{g}_i$ and $N = \exp(\mathfrak{n})$
- $\varphi|_{\mathfrak{n}}$ is a character of $\mathfrak{n}$ and hence a character ψ_N of N

Degenerate Whittaker models/coefficients, VIII

Degenerate Whittaker models/coefficients

For a representation π, locally we consider

$$\mathrm{Hom}_N(\pi, \psi_N).$$

Globally, for $\phi \in \pi$, we consider

$$\int_{N(F)\backslash N(\mathbb{A})} f(ug)\psi_N(u)\ du.$$

Nilpotent orbit attached to a representation

We say that the nilpotent orbit attached to a representation π is $\mathcal{O}$ if $\mathcal{O}$ is the maximal nilpotent orbit that supports a nonzero degenerate Whittaker model/coefficient.

The twisted doubling integrals, III

A Fourier coefficient in the orbit $((2k-1)^m 1^m)$

One can choose a nice pair (U, ψ_U) (for the group $G^{\square,k}$) in the orbit $((2k-1)^m 1^m)$ such that

$$\iota_k(\mathrm{G} \times \mathrm{G}) \subset \mathrm{Stab}(U, \psi_U).$$

(*This does not appear when* $k = 1$.)

Eisenstein series

- θ: an automorphic representation $\mathrm{GL}_{km}(\mathbb{A})$ such that (k^m) is the maximal orbit that supports a nonzero Fourier coefficient (with unique models at every local place).
- $I(s, \theta) = \mathrm{Ind}_{P(\mathbb{A})}^{G^{\square,k}(\mathbb{A})} \theta \cdot \delta_P^s$
- $f^{(s)} \in I(s, \theta)$, one attaches an Eisenstein series

$$E(f^{(s)})(g) = \sum_{\gamma \in P(F) \backslash G^{\square,k}(F)} f^{(s)}(\gamma g).$$

The twisted doubling integrals, IV

- π: irreducible cuspidal representation of $G(\mathbb{A})$
- $\xi_1 \in \pi$ and $\xi_2 \in \pi^\vee$

The global integral

We define $Z(\xi_1 \boxtimes \xi_2, f^{(s)}) =$

$$\int_{G(F)\backslash G(\mathbb{A}) \times G(F)\backslash G(\mathbb{A})} \xi_1(g_1)\xi_2(g_2) \cdot \int_{U(F)\backslash U(\mathbb{A})} E(f^{(s)})(u \cdot \iota(g_1, g_2)) \; \psi_U(u) \; du \; dg_1 \; dg_2.$$

Unfolding

$$Z(\xi_1 \boxtimes \xi_2, f^{(s)}) = \text{Euler product...}$$

Generalized Speh representations

The construction of θ decides the L-function.
Fix an integer m.

- τ: irreducible cuspidal automorphic representation of $\mathrm{GL}_k(\mathbb{A})$
- Let $\theta(m,\tau)$ be the unique irreducible quotient of

$$\tau|\cdot|^{(m-1)/2} \times \tau|\cdot|^{(m-3)/2} \times \cdots \times \tau|\cdot|^{-(m-1)/2}.$$

- Can also be realized as residues of Eisenstein series.

In other words,

$$\tau \in Irr_{cusp}(\mathrm{GL}_k(\mathbb{A})) \mapsto \theta(m,\tau) \in Irr(\mathrm{GL}_{km}(\mathbb{A})).$$

Key properties:

- the nilpotent orbit attached to $\theta(m,\tau)$ is (k^m).
- at every local place v, there is a unique model of degenerate type for $\theta(m,\tau)_v$.

L-functions

Examples

- $\tau \in Irr_{cusp}(\mathrm{GL}_k(\mathbb{A}))$, construct $\theta(m,\tau)$ of $\mathrm{GL}_{km}(\mathbb{A})$, then
$$Z(\xi_1 \boxtimes \xi_2, f^{(s)}) \approx L(s, \pi \times \tau).$$
- $\tau_i \in Irr_{cusp}(\mathrm{GL}_{k_i}(\mathbb{A}))$ for $i = 1, \cdots, \ell$. Construct
$$\theta(m,\tau_1) \boxtimes \cdots \boxtimes \theta(m,\tau_\ell),$$
then
$$Z(\xi_1 \boxtimes \xi_2, f^{(s)}) \approx L(s, \pi \times \tau_1) \cdots L(s, \pi \times \tau_\ell).$$

Covering group case, I

- $\overline{\mathrm{G}} \in \mathbf{CExt}(\mathrm{G}, \mathrm{K}_2)$.
- ϵ: a fixed embedding $\mu_n \to \mathbb{C}^\times$
- $\overline{\pi}$: irreducible ϵ-genuine cuspidal automorphic representation of $\overline{G}(\mathbb{A})$
- ϵ-genuine: μ_n acts via ϵ

Theorem

There is a global integral involving $\overline{\pi}$ that represents an Euler product.

Main step

Given $\overline{\mathrm{G}} \in \mathbf{CExt}(\mathrm{G}, \mathrm{K}_2)$, construct

$$\overline{\mathrm{G}^{\square,k}} \in \mathbf{CExt}(\mathrm{G}^{\square,k}, \mathrm{K}_2)$$

which is compatible with

$$\iota_k : \mathrm{G} \times \mathrm{G} \to \mathrm{G}^{\square,k}.$$

Covering group case, II

L-function $= ?$

This is related to the construction of θ, which is currently unknown in general.

Construction of θ, I

$Irr(\overline{\mathrm{GL}}_k^{(n)}(\mathbb{A}))$

- $\dim Wh(\tau_v) \gg 1$ for almost all $\tau \in Irr(\overline{\mathrm{GL}}_k^{(n)}(\mathbb{A}))$
- Possible: $\dim Wh(\tau_v) = 1$ for the smallest piece of a highly reducible induced representation.

Naive idea

As in the linear case,

$$\tau \in \mathrm{Irr}_{cusp}(\overline{\mathrm{GL}}_k^{(n)}(\mathbb{A})) \mapsto \theta^{(n)}(m,\tau) \in \mathrm{Irr}(\overline{\mathrm{GL}}_{kmn}^{(n)}(\mathbb{A}))$$

Hope that the nilpotent orbit attached to $\theta^{(n)}(m,\tau)$ is $((kn)^m)$ (and with unique local model at every local place).

This will not work due to the existence of *cuspidal theta representations*.

Construction of θ, II

(Conjectural picture of Toshiaki Suzuki (1998)) Assume the there is a Shimura lift

$$Sh : \overline{\mathrm{GL}}_k(\mathbb{A}) \to \mathrm{GL}_k(\mathbb{A})$$

which is local-to-global compatible. Then it can be constructed by the following diagram:

$$\begin{tikzcd}
\mathrm{Irr}_{cusp}(\overline{\mathrm{GL}}_k) \arrow[r, dashed] & \mathrm{Irr}(\overline{\mathrm{GL}}^{(n)}_{kmn}) \\
\tau \in \mathrm{Irr}_{cusp}(\mathrm{GL}_k) \arrow[u, dashed, "Sh^{-1}"] \arrow[ur, "\theta^{(n)}(m,-)"] \arrow[r, "\theta(mn,-)"] & \mathrm{Irr}(\mathrm{GL}_{kmn}) \arrow[u, "Sh^{-1}"']
\end{tikzcd}$$

Assuming this, the global zeta integral represents a $\overline{\mathrm{G}} \times \mathrm{GL}_k$ L-function:

$$Z(\xi_1 \boxtimes \xi_2, f^{(s)}) \approx L(s, \overline{\pi} \times \tau).$$

Construction of θ, III

Some evidences when $m = 1$

- $k = 1, n = 2$: Weil representation
- $k = 1, n = 3$: cubic theta representation. Can be realized by the Converse theorem or the theta correspondence using the cubic cover of the exceptional group G_2.
- $k = 2, n = 2$: (?) the theta correspondence using the double cover of the exceptional group F_4.