On Chen's conjecture for biharmonic hypersurfaces in $\mathbb{R}^5$

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Harmonic maps

Let (M^n, g) be an n-dimensional Riemannian manifold without boundary, and let (N^m, h) be another m-dimensional compact Riemannian manifold without boundary (isometrically embedded into $\mathbb{R}^L$).
Harmonic maps $\phi : (M^n, g) \to (N^m, h)$ between Riemannian manifolds are critical points of the Dirichlet energy functional

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The Euler-Lagrange equation associated to $E(\phi)$ is given by the tension field

$$\tau(\phi) = \mathrm{trace} \, \nabla \mathrm{d}\phi = 0.$$

Biharmonic maps

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For biharmonic maps, the Euler-Lagrange equation associated to $E_2(\phi)$ is:

$$\tau_2(\phi) = -\Delta\tau(\phi) - \mathrm{trace}\, R^N(d\phi, \tau(\phi))d\phi = 0, \qquad (1)$$

where Δ is the rough Laplacian of M and R^N is the curvature tensor of N.

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In Euclidean spaces, biharmonic submanifolds are defined via the geometric condition $\Delta \overrightarrow{H} = 0$, or equivalently $\Delta^2 \phi = 0$, which was originally proposed by B. Y. Chen in his pioneering work of finite type theory in the middle of 1980s.

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Theorem 1

Every biharmonic surface M^2 *in* $\mathbb{R}^3$ *is minimal.*

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Biharmonic submanifolds have received a lot of attentions and many contributions on biharmonic submanifolds have been made during last two decades.

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- Surfaces in $\mathbb{R}^3$ (Chen 1986; Jiang 1987).
- Hypersurfaces with at most two distinct principal curvatures in $\mathbb{R}^m$ (Dimitrić 1992).
- Hypersurfaces in $\mathbb{R}^4$ (Hasanis and Vlachos 1995; Defever 1998).

- Weakly convex hypersurfaces in $\mathbb{R}^m$ (Luo 2014).

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- Generic hypersufaces with irreducible principal curvature vector fields in $\mathbb{R}^m$ (Koiso and Urakawa 2018).

Although many results with extra-conditions were made, Chen's conjecture is widely open.

The Bernstein problem

The famous Bernstein problem for minimal graphs in $\mathbb{R}^{n+1}$ are: For $n \geq 2$, any entire solution $f : \mathbb{R}^n \to \mathbb{R}$ of the minimal graph equation

$$\sum_{i,j=1}^{n} \left(\delta_{ij} - \frac{f_i f_j}{1 + |\nabla f|^2} \right) f_{ij} = 0$$

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- Yes, Almgren for $n = 4$ in 1966 (Ann. Math.).
- Yes, Simons for $n \leq 7$ in 1968 (Ann. Math.).
- For $n \geq 8$, No. Bombieri-De Giorgi-Giusti in 1969 (Invent. Math.) constructed a counterexample that the Bernstein problem is not true

Bernstein problem and Chen's Conjecture

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There exists an interesting similarity between the Bernstein problem and Chen's Conjecture in the case of the graph in $\mathbb{R}^{n+1}$. Chen's conjecture for biharmonic graphs states that:

For $n \geq 2$, any entire solution $f : \mathbb{R}^n \to \mathbb{R}$ of the biharmonic graph equations:

$$\begin{cases} \Delta(\Delta f) = 0, \\ (\Delta f_k)\Delta f + 2\langle \nabla f_k, \nabla \Delta f \rangle = 0, \quad k = 1, \ldots n \end{cases}$$

is minimal; i.e.,

$$\Delta f = -\mathrm{div}(\nabla f) = 0.$$

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Fu-H (2018) showed that biharmonic hypersurfaces with constant scalar curvature in Euclidean space $\mathbb{R}^{n+1}$ for $n < 7$ are minimal.

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Fu-H (2018) showed that biharmonic hypersurfaces with constant scalar curvature in Euclidean space $\mathbb{R}^{n+1}$ for $n < 7$ are minimal.

However,the general Chen's conjecture on hypersurfaces M^n in $\mathbb{R}^{n+1}$ remains open for $n \geq 4$. (even for the biharmonic graphs)

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More precisely, in this talk, we will prove the following:

Theorem 2

(Fu-H-Zhan) Every biharmonic hypersurface in the Euclidean space $\mathbb{R}^5$ *is minimal.*

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(Fu-H-Zhan) Every biharmonic hypersurface in the Euclidean space $\mathbb{R}^5$ is minimal.

Remark: The main approach for the proof of Theorem 2 is a continuation of the program developed in joint paper with Fu (2018). However, in the paper, an extra condition of *constant scalar curvature* is assumed.

By identifying the tangent and the normal parts of the biharmonic condition (2) for hypersurfaces in $\mathbb{R}^{n+1}$, one can obtain the following characterization result:

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Lemma 3

The immersion $\phi : M^n \to \mathbb{R}^{n+1}$ of a hypersurface M^n in the Euclidean space $\mathbb{R}^{n+1}$ is biharmonic; i.e. $\Delta^2\phi = 0$, if and only if H and A satisfy

$$\begin{cases} \Delta H + H \mathrm{trace}\, A^2 = 0, \\ 2A\,\mathrm{grad} H + nH \mathrm{grad} H = 0, \end{cases} \tag{3}$$

where A is the Weingarten operator of M^n defined by $\langle h(X, Y), \xi\rangle = \langle AX, Y\rangle$ for all $X, Y \in T(M^n)$.

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The second equation of (3) shows that $\operatorname{grad} H$ is an eigenvector of the Weingarten operator A with the principal curvature $\lambda_1 = -\frac{nH}{2}$.

Note that $\sum_{i=1}^{n} \lambda_i = nH$ and $\lambda_1 = -\frac{nH}{2}$. Then we have $\sum_{i=2}^{n} \lambda_i = -3\lambda_1$.

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As $\mathrm{grad}\, H = \sum_{i=1}^{n} e_i(H) e_i$ and e_1 is parallel to the direction of $\mathrm{grad}\, H$, and with some suitable orthonormal frame $\{e_1, \ldots, e_n\}$, the Weingarten operator A of M is:

$$A = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

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Since $\nabla_{e_k} \langle e_i, e_i \rangle = 0$ and $\nabla_{e_k} \langle e_i, e_j \rangle = 0$ $(i \neq j)$, we have

$$\omega_{ki}^{i} = 0, \quad \omega_{ki}^{j} + \omega_{kj}^{i} = 0, \quad i \neq j.$$

Recall the Gauss equation

$$R(X, Y)Z = \langle AY, Z\rangle AX - \langle AX, Z\rangle AY,$$

where R is the curvature tensor of M^n and the Codazzi equation

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Then

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega^j_{ji}, \quad (5)$$

$$(\lambda_i - \lambda_j)\omega^j_{ki} = (\lambda_k - \lambda_j)\omega^j_{ik}. \quad (6)$$

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for distinct i, j, k.

$$[e_i, e_j](\lambda_1) = 0, \quad \omega_{ij}^1 = \omega_{ji}^1,\ 2 \le i, j \le n,\ i \ne j.$$

Moreover, we can prove that

Lemma 4

Let M^n be an orientable biharmonic hypersurface with non-constant mean curvature H in $\mathbb{R}^{n+1}$. Then the multiplicity of the principal curvature λ_1 $(= -nH/2)$ is one, i.e. $\lambda_j \neq \lambda_1$ for $2 \leq j \leq n$.

Moreover, we can prove that

Lemma 4

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Proof.

If $\lambda_j = \lambda_1$ for $j \neq 1$, by putting $i = 1$ in (5):

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega^j_{ji},$$

we get

$$0 = (\lambda_1 - \lambda_j)\omega^j_{j1} = e_1(\lambda_i) = e_1(\lambda_1),$$

which contradicts (4); i.e. $e_1(\lambda_1) \neq 0$. □

Some new and key lemmas

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We consider an integral curve of e_1 passing through $p = \gamma(t_0)$ as $\gamma(t)$, $t \in I$. It is easy to show that there exists a local chart $(U; t = x^1, x^2, x^3, x^4)$ around p, such that $\lambda_1(t, x^2, x^3, x^4) = \lambda_1(t)$ on the whole neighborhood of p.

Set

$$f_k = (\omega_{22}^1)^k + (\omega_{33}^1)^k + (\omega_{44}^1)^k, \quad \text{for } k = 1, \dots, 5.$$

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In the following, an interesting system of algebraic equations could be derived.

Lemma 5

With the notions f_k, the following two relations hold

$$f_1^4 - 6f_1^2 f_2 + 3f_2^2 + 8f_1 f_3 - 6f_4 = 0, \tag{7}$$

$$f_1^5 - 5f_1^3 f_2 + 5f_1^2 f_3 + 5f_2 f_3 - 6f_5 = 0. \tag{8}$$

Proof of Lemma 5

It is easy to check that

$$f_1^2 - f_2 = \Big(\sum_{i=2}^{4} \omega_{ii}^1\Big)^2 - \sum_{i=2}^{4} (\omega_{ii}^1)^2$$
$$= 2(\omega_{22}^1\omega_{33}^1 + \omega_{22}^1\omega_{44}^1 + \omega_{33}^1\omega_{44}^1), \tag{9}$$

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and

$$f_2^2 - f_4 = \Big(\sum_{i=2}^{4} \omega_{ii}^2\Big)^2 - \sum_{i=2}^{4} (\omega_{ii}^1)^4$$
$$= 2\big\{(\omega_{22}^1\omega_{33}^1)^2 + (\omega_{22}^1\omega_{44}^1)^2 + (\omega_{33}^1\omega_{44}^1)^2\big\}. \quad (10)$$

Combining (9) with (10) gives

$$
\begin{aligned}
&(f_1^2 - f_2)^2 - 2(f_2^2 - f_4) \\
&= 4(\omega_{22}^1\omega_{33}^1 + \omega_{22}^1\omega_{44}^1 + \omega_{33}^1\omega_{44}^1)^2 \\
&\quad - 4\{(\omega_{22}^1\omega_{33}^1)^2 + (\omega_{22}^1\omega_{44}^1)^2 + (\omega_{33}^1\omega_{44}^1)^2\} \\
&= 8\{(\omega_{22}^1)^2\omega_{33}^1\omega_{44}^1 + \omega_{22}^1(\omega_{33}^1)^2\omega_{44}^1 + \omega_{22}^1\omega_{33}^1(\omega_{44}^1)^2\} \\
&= 8f_1\omega_{22}^1\omega_{33}^1\omega_{44}^1. \qquad (11)
\end{aligned}
$$

Combining (9) with (10) gives

$$\begin{aligned}
&(f_1^2 - f_2)^2 - 2(f_2^2 - f_4) \\
&= 4(\omega_{22}^1\omega_{33}^1 + \omega_{22}^1\omega_{44}^1 + \omega_{33}^1\omega_{44}^1)^2 \\
&\quad - 4\{(\omega_{22}^1\omega_{33}^1)^2 + (\omega_{22}^1\omega_{44}^1)^2 + (\omega_{33}^1\omega_{44}^1)^2\} \\
&= 8\{(\omega_{22}^1)^2\omega_{33}^1\omega_{44}^1 + \omega_{22}^1(\omega_{33}^1)^2\omega_{44}^1 + \omega_{22}^1\omega_{33}^1(\omega_{44}^1)^2\} \\
&= 8f_1\omega_{22}^1\omega_{33}^1\omega_{44}^1. \qquad (11)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
f_1^3 - f_3 &= (\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1)^3 - \{(\omega_{22}^1)^3 + (\omega_{33}^1)^3 + (\omega_{44}^1)^3\} \\
&= 3\sum_{i=2}^{4}(\omega_{ii}^1)^2(f_1 - \omega_{ii}^1) + 6\omega_{22}^1\omega_{33}^1\omega_{44}^1 \\
&= 3f_1f_2 - 3f_3 + 6\omega_{22}^1\omega_{33}^1\omega_{44}^1. \qquad (12)
\end{aligned}$$

Combining (9) with (10) gives

$$\begin{aligned}
&(f_1^2 - f_2)^2 - 2(f_2^2 - f_4) \\
&= 4(\omega_{22}^1\omega_{33}^1 + \omega_{22}^1\omega_{44}^1 + \omega_{33}^1\omega_{44}^1)^2 \\
&\quad - 4\big\{(\omega_{22}^1\omega_{33}^1)^2 + (\omega_{22}^1\omega_{44}^1)^2 + (\omega_{33}^1\omega_{44}^1)^2\big\} \\
&= 8\big\{(\omega_{22}^1)^2\omega_{33}^1\omega_{44}^1 + \omega_{22}^1(\omega_{33}^1)^2\omega_{44}^1 + \omega_{22}^1\omega_{33}^1(\omega_{44}^1)^2\big\} \\
&= 8f_1\omega_{22}^1\omega_{33}^1\omega_{44}^1. \qquad (11)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
f_1^3 - f_3 &= (\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1)^3 - \big\{(\omega_{22}^1)^3 + (\omega_{33}^1)^3 + (\omega_{44}^1)^3\big\} \\
&= 3\sum_{i=2}^{4}(\omega_{ii}^1)^2(f_1 - \omega_{ii}^1) + 6\omega_{22}^1\omega_{33}^1\omega_{44}^1 \\
&= 3f_1f_2 - 3f_3 + 6\omega_{22}^1\omega_{33}^1\omega_{44}^1. \qquad (12)
\end{aligned}$$

Eliminating $\omega_{22}^1\omega_{33}^1\omega_{44}^1$ from (11) and (12), we get (7).

A direct computation shows that

$$
\begin{aligned}
f_1 f_4 &= \Big(\sum_{i=2}^{4}\omega_{ii}^1\Big)\Big(\sum_{i=2}^{4}(\omega_{ii}^1)^4\Big)\\
&= \sum_{i=2}^{4}(\omega_{ii}^1)^5 + \omega_{22}^1\Big\{(\omega_{33}^1)^4 + (\omega_{44}^1)^4\Big\}\\
&\quad + \omega_{33}^1\Big\{(\omega_{22}^1)^4 + (\omega_{44}^1)^4\Big\} + \omega_{44}^1\Big\{(\omega_{22}^1)^4 + (\omega_{33}^1)^4\Big\}\\
&= f_5 + \omega_{22}^1\left\{\left[(\omega_{33}^1)^2 + (\omega_{44}^1)^2\right]^2 - 2(\omega_{33}^1)^2(\omega_{44}^1)^2\right\}\\
&\quad + \omega_{33}^1\left\{\left[(\omega_{22}^1)^2 + (\omega_{44}^1)^2\right]^2 - 2(\omega_{22}^1)^2(\omega_{44}^1)^2\right\}\\
&\quad + \omega_{44}^1\left\{\left[(\omega_{22}^1)^2 + (\omega_{33}^1)^2\right]^2 - 2(\omega_{22}^1)^2(\omega_{33}^1)^2\right\}\\
&= f_5 + \sum_{i=2}^{4}\omega_{ii}^1\Big(f_2 - (\omega_{ii}^1)^2\Big)^2\\
&\quad - 2\omega_{22}^1\omega_{33}^1\omega_{44}^1\big(\omega_{22}^1\omega_{33}^1 + \omega_{22}^1\omega_{44}^1 + \omega_{33}^1\omega_{44}^1\big),
\end{aligned}
$$

which together with (9) yields

$$f_1 f_4 = 2f_5 + f_1 f_2^2 - 2f_2 f_3 - (f_1^2 - f_2)\omega_{22}^1\omega_{33}^1\omega_{44}^1. \qquad (13)$$

Eliminating $\omega_{22}^1\omega_{33}^1\omega_{44}^1$ from (12) and (13) again, one gets

$$6f_1 f_4 = 12f_5 + 6f_1 f_2^2 - 12f_2 f_3 - (f_1^2 - f_2)(f_1^3 - 3f_1 f_2 + 2f_3). \quad (14)$$

Moreover, eliminating the terms of f_4 from (7) and (14) gives (8). □

For simplicity, we write

$$\lambda = \lambda_1(t), \quad f_1 = T, T' = e_1(T), \quad T'' = e_1 e_1(T),$$
$$T''' = e_1 e_1 e_1(T), \quad T'''' = e_1 e_1 e_1 e_1(T).$$

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$$\lambda = \lambda_1(t), \quad f_1 = T, T' = e_1(T), \quad T'' = e_1 e_1(T),$$
$$T''' = e_1 e_1 e_1(T), \quad T'''' = e_1 e_1 e_1 e_1(T).$$

Lemma 6

$f_1, \; f_2, \; f_3, \; f_4 \;$ and f_5 *can be written as*

$$\begin{cases} f_1 = T, \\ f_2 = T' + 3\lambda^2, \\ f_3 = \frac{1}{2}T'' - \lambda^2 T + 6\lambda\lambda', \\ f_4 = \frac{1}{6}T''' - \frac{4}{3}\lambda^2 T' - \frac{5}{3}\lambda\lambda' T + 2\lambda'^2 + 4\lambda\lambda'' - 2\lambda^4, \\ f_5 = \frac{1}{24}T'''' - \frac{5}{6}\lambda^2 T'' - \frac{25}{12}\lambda\lambda' T' - \frac{1}{12}(13\lambda\lambda'' + \lambda'^2 - 12\lambda^4)T \\ \qquad +2\lambda\lambda''' + \frac{5}{3}\lambda'\lambda'' - \frac{26}{3}\lambda^3\lambda'. \end{cases} \tag{15}$$

Using the Gauss and Codazzi equations, biharmonic equations (3) can be summarized into a system of $2n-1$ differential equations as follows (Fu and H, Pacific J. Math. 2018):

Using the Gauss and Codazzi equations, biharmonic equations (3) can be summarized into a system of $2n-1$ differential equations as follows (Fu and H, Pacific J. Math. 2018):

Lemma 7

Assume that H is non-constant. Then the smooth real-valued principal curvature functions λ_i and the coefficients of connection ω_{ii}^1 $(i = 2, \ldots, n)$ satisfy the following differential equations:

$$e_1 e_1(\lambda_1) = e_1(\lambda_1)\Big(\sum_{i=2}^{n} \omega_{ii}^1\Big) + \lambda_1 S, \tag{16}$$

$$e_1(\lambda_i) = \lambda_i \omega_{ii}^1 - \lambda_1 \omega_{ii}^1, \tag{17}$$

$$e_1(\omega_{ii}^1) = (\omega_{ii}^1)^2 + \lambda_1 \lambda_i, \tag{18}$$

where $\lambda_1 = -nH/2$, S is the squared length of the second fundamental form h of M and $e_1 = \mathrm{grad}\, H/|\mathrm{grad}\, H|$.

Proof of Lemma 6

Since $e_1(\lambda) \neq 0$, λ is not constant. From (16), one has

$$f_1 = \frac{e_1 e_1(\lambda) - \lambda S}{e_1(\lambda)} = \frac{\lambda''}{\lambda'} - \frac{\lambda}{\lambda'} S =: T. \quad (19)$$

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Taking the sum of i from 2 to 4 in (18) and (17) respectively, using $\lambda_2 + \lambda_3 + \lambda_4 = -3\lambda_1$ we have

$$f_2 = 3\lambda^2 + e_1(f_1) = T' + 3\lambda^2, \tag{20}$$

$$g_1 := \sum_{i=2}^{4} \lambda_i \omega_{ii}^1 = \lambda T - 3e_1(\lambda) = \lambda T - 3\lambda'. \tag{21}$$

Proof of Lemma 6

Since $e_1(\lambda) \neq 0$, λ is not constant. From (16), one has

$$f_1 = \frac{e_1 e_1(\lambda) - \lambda S}{e_1(\lambda)} = \frac{\lambda''}{\lambda'} - \frac{\lambda}{\lambda'} S =: T. \tag{19}$$

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$$g_1 := \sum_{i=2}^{4} \lambda_i \omega_{ii}^1 = \lambda T - 3e_1(\lambda) = \lambda T - 3\lambda'. \tag{21}$$

Multiplying ω_{ii}^1 on both sides of equation (18), we have

$$\frac{1}{2} e_1\left((\omega_{ii}^1)^2\right) = (\omega_{ii}^1)^3 + \lambda\lambda_i\omega_{ii}^1.$$

Taking the sum of i in the above equation gives

$$f_3 = \frac{1}{2}e_1(f_2) - \lambda g_1 = \frac{1}{2}T'' - \lambda^2 T + 6\lambda\lambda'. \tag{22}$$

Taking the sum of i in the above equation gives

$$f_3 = \frac{1}{2}e_1(f_2) - \lambda g_1 = \frac{1}{2}T'' - \lambda^2 T + 6\lambda\lambda'. \tag{22}$$

Differentiating (21) along e_1, using (17) and (18) we have

$$e_1(g_1) = 2\sum_{i=2}^{4}\lambda_i\left(\omega_{ii}^1\right)^2 + \lambda\sum_{i=2}^{4}\lambda_i^2 - \lambda\sum_{i=2}^{4}\left(\omega_{ii}^1\right)^2. \tag{23}$$

Taking the sum of i in the above equation gives

$$f_3 = \frac{1}{2}e_1(f_2) - \lambda g_1 = \frac{1}{2}T'' - \lambda^2 T + 6\lambda\lambda'. \tag{22}$$

Differentiating (21) along e_1, using (17) and (18) we have

$$e_1(g_1) = 2\sum_{i=2}^{4} \lambda_i(\omega_{ii}^1)^2 + \lambda\sum_{i=2}^{4}\lambda_i^2 - \lambda\sum_{i=2}^{4}(\omega_{ii}^1)^2. \tag{23}$$

Hence, it follows from (19); i.e. $\lambda S = \lambda'' - \lambda' T$, that

$$\begin{aligned} g_2 := \sum_{i=2}^{4}\lambda_i(\omega_{ii}^1)^2 =& \frac{1}{2}\{e_1(g_1) - \lambda(S - \lambda^2) + \lambda f_2\} \\ =& \frac{1}{2}\{e_1(g_1) - \lambda'' + \lambda' T + \lambda^3 + \lambda f_2\}. \end{aligned}$$

Using (20) and (21), the above expression reduces to

$$g_2 = \lambda T' + \lambda' T - 2\lambda'' + 2\lambda^3. \tag{24}$$

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Multiplying $(\omega_{ii}^1)^2$ on both sides of equation (18), we have

$$\frac{1}{3} e_1\big((\omega_{ii}^1)^3\big) = (\omega_{ii}^1)^4 + \lambda\lambda_i(\omega_{ii}^1)^2.$$

Using (20) and (21), the above expression reduces to

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Multiplying $(\omega_{ii}^1)^2$ on both sides of equation (18), we have

$$\frac{1}{3} e_1\big((\omega_{ii}^1)^3\big) = (\omega_{ii}^1)^4 + \lambda\lambda_i(\omega_{ii}^1)^2.$$

Taking the sum of i from 2 to 4 in the above equation, we obtain

$$\begin{aligned} f_4 =& \frac{1}{3} e_1(f_3) - \lambda g_2 \\ =& \frac{1}{6} T''' - \frac{4}{3}\lambda^2 T' - \frac{5}{3}\lambda\lambda' T + 2\lambda'^2 + 4\lambda\lambda'' - 2\lambda^4. \end{aligned} \tag{25}$$

Using (20) and (21), the above expression reduces to

$$g_2 = \lambda T' + \lambda' T - 2\lambda'' + 2\lambda^3. \tag{24}$$

Multiplying $(\omega_{ii}^1)^2$ on both sides of equation (18), we have

$$\frac{1}{3} e_1\big((\omega_{ii}^1)^3\big) = (\omega_{ii}^1)^4 + \lambda\lambda_i(\omega_{ii}^1)^2.$$

Taking the sum of i from 2 to 4 in the above equation, we obtain

$$\begin{aligned} f_4 =& \frac{1}{3} e_1(f_3) - \lambda g_2 \\ =& \frac{1}{6} T''' - \frac{4}{3}\lambda^2 T' - \frac{5}{3}\lambda\lambda' T + 2\lambda'^2 + 4\lambda\lambda'' - 2\lambda^4. \end{aligned} \tag{25}$$

Multiplying equation (17) by λ_i gives

$$\lambda_i^2 \omega_{ii}^1 = \frac{1}{2} e_1(\lambda_i^2) + \lambda\lambda_i\omega_{ii}^1,$$

which yields

$$
\begin{aligned}
g_3 := \sum_{i=2}^{4} \lambda_i^2 \omega_{ii}^1 &= \frac{1}{2} e_1 (S - \lambda^2) + \lambda g_1 \\
&= \frac{1}{2} \Big(\frac{\lambda'' - \lambda' T}{\lambda} - \lambda^2 \Big)' + \lambda g_1 \\
&= -\frac{\lambda'}{2\lambda} T' + \Big(\lambda^2 - \frac{\lambda'' \lambda - \lambda'^2}{2\lambda^2} \Big) T \\
&- 4\lambda\lambda' + \frac{\lambda''' \lambda - \lambda'' \lambda'}{2\lambda^2}.
\end{aligned} \tag{26}
$$

which yields

$$g_3 := \sum_{i=2}^{4} \lambda_i^2 \omega_{ii}^1 = \frac{1}{2} e_1(S - \lambda^2) + \lambda g_1$$
$$= \frac{1}{2}\Big(\frac{\lambda'' - \lambda' T}{\lambda} - \lambda^2\Big)' + \lambda g_1$$
$$= -\frac{\lambda'}{2\lambda} T' + \Big(\lambda^2 - \frac{\lambda''\lambda - \lambda'^2}{2\lambda^2}\Big) T$$
$$- 4\lambda\lambda' + \frac{\lambda'''\lambda - \lambda''\lambda'}{2\lambda^2}. \quad (26)$$

Differentiating (24) with respect to e_1 and using (17) and (18), we have

$$e_1(g_2) = 3\sum_{i=2}^{4} \lambda_i \big(\omega_{ii}^1\big)^3 - \lambda \sum_{i=2}^{4} \big(\omega_{ii}^1\big)^3 + 2\lambda \sum_{i=2}^{4} \lambda_i^2 \omega_{ii}^1,$$

which leads to

$$g_4 := \sum_{i=2}^{4} \lambda_i \left(\omega_{ii}^1\right)^3 = \frac{1}{3}\left(e_1(g_2) + \lambda f_3 - 2\lambda g_3\right)$$
$$= \frac{1}{2}\lambda T'' + \lambda' T' + \frac{1}{3}(2\lambda'' - 3\lambda^3 - \frac{\lambda'^2}{\lambda})T$$
$$- \lambda''' + \frac{20}{3}\lambda^2\lambda' + \frac{\lambda''\lambda'}{3\lambda}. \qquad (27)$$

which leads to

$$g_4 := \sum_{i=2}^{4} \lambda_i \big(\omega_{ii}^1\big)^3 = \frac{1}{3}\big(e_1(g_2) + \lambda f_3 - 2\lambda g_3\big)$$
$$= \frac{1}{2}\lambda T'' + \lambda' T' + \frac{1}{3}(2\lambda'' - 3\lambda^3 - \frac{\lambda'^2}{\lambda})T$$
$$- \lambda''' + \frac{20}{3}\lambda^2\lambda' + \frac{\lambda''\lambda'}{3\lambda}. \qquad (27)$$

Multiplying $(\omega_{ii}^1)^3$ on both sides of equation (18), we have

$$\frac{1}{4} e_1\big((\omega_{ii}^1)^4\big) = (\omega_{ii}^1)^5 + \lambda\lambda_i(\omega_{ii}^1)^3.$$

which leads to

$$
\begin{aligned}
g_4 :=\sum_{i=2}^{4}\lambda_i\big(\omega_{ii}^1\big)^3 &= \frac{1}{3}\big(e_1(g_2)+\lambda f_3-2\lambda g_3\big)\\
&=\frac{1}{2}\lambda T''+\lambda' T'+\frac{1}{3}(2\lambda''-3\lambda^3-\frac{\lambda'^2}{\lambda})T\\
&\quad-\lambda'''+\frac{20}{3}\lambda^2\lambda'+\frac{\lambda''\lambda'}{3\lambda}. \qquad (27)
\end{aligned}
$$

Multiplying $(\omega_{ii}^1)^3$ on both sides of equation (18), we have

$$
\frac{1}{4}e_1\big((\omega_{ii}^1)^4\big)=(\omega_{ii}^1)^5+\lambda\lambda_i(\omega_{ii}^1)^3.
$$

After taking the sum of i in the above equation, we have

$$
\begin{aligned}
f_5 =&\frac{1}{4}e_1(f_4)-\lambda g_4\\
=&\frac{1}{24}T''''-\frac{5}{6}\lambda^2T''-\frac{25}{12}\lambda\lambda' T'-\frac{1}{12}(13\lambda\lambda''+\lambda'^2-12\lambda^4)T\\
&+2\lambda\lambda'''+\frac{5}{3}\lambda'\lambda''-\frac{26}{3}\lambda^3\lambda'. \qquad (28)
\end{aligned}
$$

Recall from Lemma 5 that

$$\begin{aligned} f_1^4 - 6f_1^2 f_2 + 3f_2^2 + 8f_1 f_3 - 6f_4 = 0, \\ f_1^5 - 5f_1^3 f_2 + 5f_1^2 f_3 + 5f_2 f_3 - 6f_5 = 0 \end{aligned}$$

and Lemma 6 that

$$\begin{cases} f_1 = T, \\ f_2 = T' + 3\lambda^2, \\ f_3 = \frac{1}{2}T'' - \lambda^2 T + 6\lambda\lambda', \\ f_4 = \frac{1}{6}T''' - \frac{4}{3}\lambda^2 T' - \frac{5}{3}\lambda\lambda' T + 2\lambda'^2 + 4\lambda\lambda'' - 2\lambda^4, \\ f_5 = \frac{1}{24}T'''' - \frac{5}{6}\lambda^2 T'' - \frac{25}{12}\lambda\lambda' T' - \frac{1}{12}(13\lambda\lambda'' + \lambda'^2 - 12\lambda^4)T \\ \qquad +2\lambda\lambda''' + \frac{5}{3}\lambda'\lambda'' - \frac{26}{3}\lambda^3\lambda'. \end{cases}$$

Recall from Lemma 5 that

$$f_1^4 - 6f_1^2 f_2 + 3f_2^2 + 8f_1 f_3 - 6f_4 = 0,$$
$$f_1^5 - 5f_1^3 f_2 + 5f_1^2 f_3 + 5f_2 f_3 - 6f_5 = 0$$

and Lemma 6 that

$$\begin{cases} f_1 = T, \\ f_2 = T' + 3\lambda^2, \\ f_3 = \frac{1}{2}T'' - \lambda^2 T + 6\lambda\lambda', \\ f_4 = \frac{1}{6}T''' - \frac{4}{3}\lambda^2 T' - \frac{5}{3}\lambda\lambda' T + 2\lambda'^2 + 4\lambda\lambda'' - 2\lambda^4, \\ f_5 = \frac{1}{24}T'''' - \frac{5}{6}\lambda^2 T'' - \frac{25}{12}\lambda\lambda' T' - \frac{1}{12}(13\lambda\lambda'' + \lambda'^2 - 12\lambda^4)T \\ \qquad +2\lambda\lambda''' + \frac{5}{3}\lambda'\lambda'' - \frac{26}{3}\lambda^3\lambda'. \end{cases}$$

Lemma 8

Let M^4 is an orientable biharmonic hypersurface with simple distinct principal curvatures in $\mathbb{R}^5$. Then the function T depends only on the variable t.

A sketch proof of Lemma 8

Substituting (15) into (7) and (8) yields

$$- T''' + 4TT'' + 3T'^2 + (-6T^2 + 26\lambda^2)T' + (T^4 - 26\lambda^2 T^2 + 58\lambda\lambda' T) + 39\lambda^4 - 24\lambda\lambda'' - 12\lambda'^2 = 0, \tag{29}$$

and

$$- T'''' + 10T'T'' + (10T^2 + 50\lambda^2)T'' - (20T^3 + 20\lambda^2 T - 170\lambda\lambda')T' + (4T^5 - 80\lambda^2 T^3 + 120\lambda\lambda' T^2 - 84\lambda^4 T + 26\lambda\lambda'' T + 2\lambda'^2 T) + 568\lambda^3\lambda' - 48\lambda\lambda''' - 40\lambda'\lambda'' = 0. \tag{30}$$

A sketch proof of Lemma 8

Substituting (15) into (7) and (8) yields

$$\begin{aligned} &-T''' + 4TT'' + 3T'^2 + (-6T^2 + 26\lambda^2)T' + (T^4 - 26\lambda^2 T^2 \\ &+ 58\lambda\lambda' T) + 39\lambda^4 - 24\lambda\lambda'' - 12\lambda'^2 = 0, \end{aligned} \tag{29}$$

and

$$\begin{aligned} &-T'''' + 10T'T'' + (10T^2 + 50\lambda^2)T'' - (20T^3 + 20\lambda^2 T \\ &- 170\lambda\lambda')T' + (4T^5 - 80\lambda^2 T^3 + 120\lambda\lambda' T^2 - 84\lambda^4 T + 26\lambda\lambda'' T \\ &+ 2\lambda'^2 T) + 568\lambda^3\lambda' - 48\lambda\lambda''' - 40\lambda'\lambda'' = 0. \end{aligned} \tag{30}$$

Notice that Equations (29) and (30) are entirely different differential equations with respect to T.

A sketch proof of Lemma 8

Substituting (15) into (7) and (8) yields

$$
\begin{aligned}
&-T''' + 4TT'' + 3T'^2 + (-6T^2 + 26\lambda^2)T' + (T^4 - 26\lambda^2 T^2 \\
&+ 58\lambda\lambda' T) + 39\lambda^4 - 24\lambda\lambda'' - 12\lambda'^2 = 0, \qquad (29)
\end{aligned}
$$

and

$$
\begin{aligned}
&-T'''' + 10T'T'' + (10T^2 + 50\lambda^2)T'' - (20T^3 + 20\lambda^2 T \\
&- 170\lambda\lambda')T' + (4T^5 - 80\lambda^2 T^3 + 120\lambda\lambda' T^2 - 84\lambda^4 T + 26\lambda\lambda'' T \\
&+ 2\lambda'^2 T) + 568\lambda^3\lambda' - 48\lambda\lambda''' - 40\lambda'\lambda'' = 0. \qquad (30)
\end{aligned}
$$

Notice that Equations (29) and (30) are entirely different differential equations with respect to T.

we intend to eliminate the terms of T'''', T''', T'', T' term by term and derive a non-trivial polynomial equation of T with the coefficients depending only on the variable t.

In fact, differentiating (29) with respect to e_1, we have

$$
\begin{aligned}
&- T'''' + 4TT''' + 10T'T'' + (-6T^2 + 26\lambda^2)T'' - 12TT'^2 \\
&+ (4T^3 - 52\lambda^2 T + 110\lambda\lambda')T' + (-52\lambda\lambda' T^2 + 58\lambda\lambda'' T \\
&+ 58\lambda'^2 T) + 156\lambda^3\lambda' - 48\lambda'\lambda'' - 24\lambda\lambda''' = 0. \qquad (31)
\end{aligned}
$$

In fact, differentiating (29) with respect to e_1, we have

$$
\begin{aligned}
&- T'''' + 4TT''' + 10T'T'' + (-6T^2 + 26\lambda^2)T'' - 12TT'^2 \\
&+ (4T^3 - 52\lambda^2 T + 110\lambda\lambda')T' + (-52\lambda\lambda' T^2 + 58\lambda\lambda'' T \\
&+ 58\lambda'^2 T) + 156\lambda^3\lambda' - 48\lambda'\lambda'' - 24\lambda\lambda''' = 0. \qquad (31)
\end{aligned}
$$

Eliminating the terms on T'''' in (30)-(31), we get

$$
\begin{aligned}
&4TT''' - (16T^2 + 24\lambda^2)T'' - 12TT'^2 + (24T^3 - 32\lambda^2 T - 60\lambda\lambda')T' \\
&+ (-4T^5 + 80\lambda^2 T^3 - 172\lambda\lambda' T^2 + 84\lambda^4 T + 32\lambda\lambda'' T + 56\lambda'^2 T) \\
&- 412\lambda^3\lambda' - 8\lambda'\lambda'' + 24\lambda\lambda''' = 0. \qquad (32)
\end{aligned}
$$

Moreover, we can eliminate the terms on T''' of (29) and (32). We obtain

$$
\begin{aligned}
&-6\lambda^2 T'' + (18\lambda^2 T - 15\lambda\lambda')T' \\
&+ (-6\lambda^2 T^3 + 15\lambda\lambda' T^2 + 60\lambda^4 T - 16\lambda\lambda'' T + 2\lambda'^2 T) \\
&- 103\lambda^3\lambda' - 2\lambda'\lambda'' + 6\lambda\lambda''' = 0.
\end{aligned} \tag{33}
$$

Moreover, we can eliminate the terms on T''' of (29) and (32). We obtain

$$
\begin{aligned}
&-6\lambda^2 T'' + (18\lambda^2 T - 15\lambda\lambda')T' \\
&+ (-6\lambda^2 T^3 + 15\lambda\lambda' T^2 + 60\lambda^4 T - 16\lambda\lambda'' T + 2\lambda'^2 T) \\
&- 103\lambda^3\lambda' - 2\lambda'\lambda'' + 6\lambda\lambda''' = 0. \qquad (33)
\end{aligned}
$$

Differentiating the above equation along e_1, one sees

$$
\begin{aligned}
&-6\lambda^2 T''' + (18\lambda^2 T - 27\lambda\lambda')T'' + 18\lambda^2 T'^2 \\
&+ (-18\lambda^2 T^2 + 66\lambda\lambda' T + 60\lambda^4 - 13\lambda'^2 - 31\lambda\lambda'')T' \\
&+ (-12\lambda\lambda' T^3 + 15\lambda'^2 T^2 + 15\lambda\lambda'' T^2 + 240\lambda^3\lambda' T \\
&- 12\lambda'\lambda'' T - 16\lambda\lambda''' T) - 309\lambda^2\lambda'^2 - 103\lambda^3\lambda'' \\
&- 2\lambda''^2 + 4\lambda'\lambda''' + 6\lambda\lambda'''' = 0. \qquad (34)
\end{aligned}
$$

Note that both equations (29)

$$-T''' + 4TT'' + 3T'^2 + (-6T^2 + 26\lambda^2)T' + (T^4 - 26\lambda^2 T^2 + 58\lambda\lambda' T) + 39\lambda^4 - 24\lambda\lambda'' - 12\lambda'^2 = 0,$$

and (32)

$$4TT''' - (16T^2 + 24\lambda^2)T'' - 12TT'^2 + (24T^3 - 32\lambda^2 T - 60\lambda\lambda')T' + (-4T^5 + 80\lambda^2 T^3 - 172\lambda\lambda' T^2 + 84\lambda^4 T + 32\lambda\lambda'' T + 56\lambda'^2 T) - 412\lambda^3\lambda' - 8\lambda'\lambda'' + 24\lambda\lambda''' = 0.$$

have a non-zero term of T^4, but (34)

$$-6\lambda^2 T''' + (18\lambda^2 T - 27\lambda\lambda')T'' + 18\lambda^2 T'^2 + (-18\lambda^2 T^2 + 66\lambda\lambda' T + 60\lambda^4 - 13\lambda'^2 - 31\lambda\lambda'')T' + (-12\lambda\lambda' T^3 + 15\lambda'^2 T^2 + 15\lambda\lambda'' T^2 + 240\lambda^3\lambda' T - 12\lambda'\lambda'' T - 16\lambda\lambda''' T) - 309\lambda^2\lambda'^2 - 103\lambda^3\lambda'' - 2\lambda''^2 + 4\lambda'\lambda''' + 6\lambda\lambda'''' = 0.$$

does not involve any term of T^4.

Therefore, we conclude that (34) are entirely different from equations (29) and (32), which are third-order differential equations with respect to T.
Next, we may eliminate the terms of T''', T'', T' and derive a non-trivial equation of T.

Therefore, we conclude that the function T depends only on the variable t. □

Observe easily from (17) and (18) that

$$\lambda_i \neq \lambda_j \quad \Leftrightarrow \quad \omega_{ii}^1 \neq \omega_{jj}^1.$$

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According to Lemmas 6 and 8, (15) implies that f_k for $k = 1, \ldots, 5$ depend only on the variable t, that is, $e_i(f_k) = 0$ for $2 \leq i \leq 4$.

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Moreover, we can prove that

Lemma 9

Let M^4 be an orientable biharmonic hypersurface with non-constant mean curvature in $\mathbb{R}^5$. Then $e_i(\lambda_j) = 0$ for $2 \leq i, j \leq 4$, that is, all principal curvature λ_i depend only on the variable t.

Proof of Lemma 9

Differentiating both sides of equations $f_k = \sum_{i=2}^{4}(\omega_{ii}^1)^k$ for $k = 1, 2, 3$ with respect to e_i $(2 \leq i \leq 4)$,

Proof of Lemma 9

Differentiating both sides of equations $f_k = \sum_{i=2}^{4} (\omega_{ii}^1)^k$ for $k = 1, 2, 3$ with respect to e_i $(2 \leq i \leq 4)$, we obtain

$$\begin{cases} e_i(\omega_{22}^1) + e_i(\omega_{33}^1) + e_i(\omega_{44}^1) = 0, \\ \omega_{22}^1 e_i(\omega_{22}^1) + \omega_{33}^1 e_i(\omega_{33}^1) + \omega_{44}^1 e_i(\omega_{44}^1) = 0, \\ (\omega_{22}^1)^2 e_i(\omega_{22}^1) + (\omega_{33}^1)^2 e_i(\omega_{33}^1) + (\omega_{44}^1)^2 e_i(\omega_{44}^1) = 0. \end{cases} \tag{35}$$

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Since ω_{22}^1, ω_{33}^1, ω_{44}^1 are mutually different and the determinant of

the coefficient matrix of (35) is the Vandermonde determinant with order 3,

it follows that

$$\begin{vmatrix} 1 & 1 & 1 \\ \omega_{22}^1 & \omega_{33}^1 & \omega_{44}^1 \\ (\omega_{22}^1)^2 & (\omega_{33}^1)^2 & (\omega_{44}^1)^2 \end{vmatrix} = (\omega_{44}^1 - \omega_{33}^1)(\omega_{44}^1 - \omega_{22}^1)(\omega_{33}^1 - \omega_{22}^1) \neq 0.$$

According to Cramer's rule in linear algebra, one gets $e_i(\omega_{22}^1) = e_i(\omega_{33}^1) = e_i(\omega_{44}^1) = 0$.

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Furthermore, for $j = 2, 3, 4$ by considering

$$e_i e_1(\omega_{jj}^1) - e_1 e_i(\omega_{jj}^1) = [e_i, e_1](\omega_{jj}^1) = \sum_{l=2}^{4}(\omega_{i1}^l - \omega_{1i}^l)e_l(\omega_{jj}^1),$$

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we get

$$e_i e_1(\omega_{jj}^1) = 0.$$

Differentiating $e_1(\omega_{jj}^1) = (\omega_{jj}^1)^2 + \lambda_1\lambda_j$ with respect to e_i and taking into account the above equation and $e_i(\omega_{jj}^1) = 0$, we derive

$$e_i(\lambda_j) = 0$$

for any $1 \leq j \leq 4$ and $2 \leq i \leq 4$.

Therefore, we complete a proof of Lemma 9.

We recall some relations concerning the coefficients of connection and principal curvature functions verified by Fu and Hong (Pacific J. Math. 2018).

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Lemma 10

For three distinct principal curvatures λ_i, λ_j and λ_k $(2 \leq i, j, k \leq 4)$, we have the following relations:

$$\omega_{23}^4(\lambda_3 - \lambda_4) = \omega_{32}^4(\lambda_2 - \lambda_4) = \omega_{43}^2(\lambda_3 - \lambda_2), \tag{36}$$

$$\omega_{23}^4\omega_{32}^4 + \omega_{34}^2\omega_{43}^2 + \omega_{24}^3\omega_{42}^3 = 0, \tag{37}$$

$$\omega_{23}^4(\omega_{33}^1 - \omega_{44}^1) = \omega_{32}^4(\omega_{22}^1 - \omega_{44}^1) = \omega_{43}^2(\omega_{33}^1 - \omega_{22}^1). \tag{38}$$

Lemma 11

Under the assumptions as above, we have

$$\omega_{22}^{1}\omega_{33}^{1} - 2\omega_{23}^{4}\omega_{32}^{4} = -\lambda_2\lambda_3, \tag{39}$$

$$\omega_{22}^{1}\omega_{44}^{1} - 2\omega_{24}^{3}\omega_{42}^{3} = -\lambda_2\lambda_4, \tag{40}$$

$$\omega_{33}^{1}\omega_{44}^{1} - 2\omega_{34}^{2}\omega_{43}^{2} = -\lambda_3\lambda_4. \tag{41}$$

We used the Gauss equation

$$\langle R(e_i, e_j)e_i, e_i\rangle = -\lambda_i\lambda_j.$$

Outline proof of Theorem 2

If the mean curvature H is constant, a result due to Oniciuc (2002) shows that any biharmonic hypersurface with constant mean curvature in $\mathbb{R}^{n+1}$ is minimal.

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Hence, we assume that the mean curvature H is not constant. According to Lemma 10, i.e.

$$\omega_{23}^4(\lambda_3 - \lambda_4) = \omega_{32}^4(\lambda_2 - \lambda_4) = \omega_{43}^2(\lambda_3 - \lambda_2)$$

we distinguish the following two cases:

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we distinguish the following two cases:

Case A. $\omega_{23}^4 \neq 0$, $\omega_{32}^4 \neq 0$, and $\omega_{43}^2 \neq 0$.

Case B. $\omega_{23}^4 = 0$, $\omega_{32}^4 = 0$, and $\omega_{43}^2 = 0$.

Next we only outline a proof for Case A.

Assume that $\omega_{23}^4 \neq 0$, $\omega_{32}^4 \neq 0$, and $\omega_{43}^2 \neq 0$.
Then equations (36) and (38) reduce to

$$\frac{\omega_{33}^1 - \omega_{44}^1}{\lambda_3 - \lambda_4} = \frac{\omega_{33}^1 - \omega_{22}^1}{\lambda_3 - \lambda_2} = \frac{\omega_{44}^1 - \omega_{22}^1}{\lambda_4 - \lambda_2} = \alpha,$$

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and hence there exists a smooth functions β depending on t such that

$$\omega_{ii}^1 = \alpha\lambda_i + \beta \qquad (42)$$

for $i = 2, 3, 4$.

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and hence there exists a smooth functions β depending on t such that

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for $i = 2, 3, 4$.
From Lemma 5, we know

$$e_1 e_1(\lambda_1) = e_1(\lambda_1)\left(\sum_{i=2}^{n} \omega_{ii}^1\right) + \lambda_1 S, \tag{43}$$

$$e_1(\lambda_i) = \lambda_i \omega_{ii}^1 - \lambda_1 \omega_{ii}^1, \tag{44}$$

$$e_1(\omega_{ii}^1) = (\omega_{ii}^1)^2 + \lambda_1 \lambda_i. \tag{45}$$

Differentiating with respect to e_1 on both sides of equation (42), using (44) and (45) we get

$$e_1(\alpha) = \lambda_1(\alpha^2 + 1) + \alpha\beta, \quad (46)$$

$$e_1(\beta) = \beta(\alpha\lambda_1 + \beta). \quad (47)$$

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Taking a sum on i in (42), one has

$$\sum_{i=2}^{4} \omega_{ii}^{1} = -3\alpha\lambda_1 + 3\beta. \tag{48}$$

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Taking into account (37), equations (39), (40) and (41) lead to

$$\omega_{22}^1\omega_{33}^1 + \omega_{22}^1\omega_{44}^1 + \omega_{33}^1\omega_{44}^1 = -\lambda_2\lambda_3 - \lambda_2\lambda_4 - \lambda_3\lambda_4,$$

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which combining with (42) further reduces to

$$(1+\alpha^2)(\lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4) + 2\alpha\beta(\lambda_2 + \lambda_3 + \lambda_4) + 3\beta^2 = 0. \tag{49}$$

Since $S - \lambda_1^2 = \sum_{i=2}^4 \lambda_i^2$ and $-3\lambda_1 = \sum_{i=2}^4 \lambda_i$, it follows from (49) that

$$(1+\alpha^2)S = 10(1+\alpha^2)\lambda_1^2 - 12\alpha\beta\lambda_1 + 6\beta^2. \tag{50}$$

Since $S-\lambda_1^2=\sum_{i=2}^4\lambda_i^2$ and $-3\lambda_1=\sum_{i=2}^4\lambda_i$, it follows from (49) that

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Moreover, differentiating $-3\lambda_1=\sum_{i=2}^4\lambda_i$ with respect to e_1 and using (17) and (42), we get

$$\begin{aligned}-3e_1(\lambda_1)&=\sum_{i=2}^{4}(\lambda_i-\lambda_1)\omega_{ii}^1.\\&=\alpha(2\lambda_1^2+S)-6\beta\lambda_1.\end{aligned} \tag{51}$$

Substituting (48) into (16) gives

$$e_1e_1(\lambda_1)=3(-\alpha\lambda_1+\beta)e_1(\lambda_1)+\lambda_1S. \tag{52}$$

Since $S - \lambda_1^2 = \sum_{i=2}^4 \lambda_i^2$ and $-3\lambda_1 = \sum_{i=2}^4 \lambda_i$, it follows from (49) that

$$(1+\alpha^2)S = 10(1+\alpha^2)\lambda_1^2 - 12\alpha\beta\lambda_1 + 6\beta^2. \tag{50}$$

Moreover, differentiating $-3\lambda_1 = \sum_{i=2}^4 \lambda_i$ with respect to e_1 and using (17) and (42), we get

$$\begin{aligned} -3e_1(\lambda_1) &= \sum_{i=2}^{4} (\lambda_i - \lambda_1)\omega_{ii}^1. \\ &= \alpha(2\lambda_1^2 + S) - 6\beta\lambda_1. \end{aligned} \tag{51}$$

Substituting (48) into (16) gives

$$e_1 e_1(\lambda_1) = 3(-\alpha\lambda_1 + \beta)e_1(\lambda_1) + \lambda_1 S. \tag{52}$$

In summaries, we have

$$e_1(\alpha) = \lambda_1(\alpha^2 + 1) + \alpha\beta,$$

$$e_1(\beta) = \beta(\alpha\lambda_1 + \beta),$$

$$(1 + \alpha^2)S = 10(1 + \alpha^2)\lambda_1^2 - 12\alpha\beta\lambda_1 + 6\beta^2,$$

$$-3e_1(\lambda_1) = \alpha(2\lambda_1^2 + S) - 6\beta\lambda_1,$$

$$e_1 e_1(\lambda_1) = 3(-\alpha\lambda_1 + \beta)e_1(\lambda_1) + \lambda_1 S.$$

By using (50) and (51), we can eliminate S and get

$$e_1(\lambda_1) = -\frac{1}{1+\alpha^2}(4\lambda_1^2\alpha^3 - 6\lambda_1\alpha^2\beta + 2\alpha\beta^2 + 4\lambda_1^2\alpha - 2\lambda_1\beta). \tag{53}$$

On the other hand, differentiating (51) with respect to e_1, it follows from (46) and (47) that

$$\begin{aligned} -3e_1 e_1(\lambda_1) =& (4\alpha\lambda_1 - 6\beta)e_1(\lambda_1) + \alpha e_1(S) \\ &+ (2\lambda_1^2 + S)\{\lambda_1(\alpha^2+1) + \alpha\beta\} - 6\beta\lambda_1(\alpha\lambda_1 + \beta). \end{aligned} \tag{54}$$

Differentiating (50) with respect to e_1 and using (46)-(47) one has

$$\begin{aligned}(1+\alpha^2)e_1(S) =& 4\big\{5(1+\alpha^2)\lambda_1 - 3\alpha\beta\big\}e_1(\lambda_1)\\ &+ 2(10\alpha\lambda_1^2 - \alpha S - 6\beta\lambda_1)\big\{\lambda_1(\alpha^2+1)+\alpha\beta\big\}\\ &+ 12\beta(\beta-\alpha\lambda_1)(\alpha\lambda_1+\beta).\end{aligned} \tag{55}$$

Eliminating the terms of $e_1e_1(\lambda_1)$ between (54) and (52) yields

$$\begin{aligned}(-5\alpha\lambda_1+3\beta)e_1(\lambda_1)+\alpha e_1(S)+(2\lambda_1^2+S)\big\{\lambda_1(\alpha^2+1)+\alpha\beta\big\}\\ -6\beta\lambda_1(\alpha\lambda_1+\beta)+3\lambda_1 S = 0.\end{aligned} \tag{56}$$

Combining (56) with (51) we may eliminate $e_1(\lambda_1)$ and hence

$$\begin{aligned} 3\alpha e_1(S) =& (-5\alpha\lambda_1 + 3\beta)\{(2\lambda_1^2 + S)\alpha - 6\beta\lambda_1\} \\ &- 3(2\lambda_1^2 + S)\{\lambda_1(\alpha^2 + 1) + \alpha\beta\} \\ &+ 18\beta\lambda_1(\alpha\lambda_1 + \beta) - 9\lambda_1 S. \end{aligned} \tag{57}$$

Also, combining (55) with (51), we eliminate $e_1(\lambda_1)$ to obtain

$$\begin{aligned} 3(1+\alpha^2)e_1(S) =& 4\{5(1+\alpha^2)\lambda_1 - 3\alpha\beta\}\{6\beta\lambda_1 - (2\lambda_1^2 + S)\alpha\} \\ &+ 6(10\alpha\lambda_1^2 - \alpha S - 6\beta\lambda_1)\{\lambda_1(\alpha^2+1) + \alpha\beta\} \\ &+ 36\beta(\beta - \alpha\lambda_1)(\alpha\lambda_1 + \beta). \end{aligned} \tag{58}$$

Eliminating the terms of $e_1(S)$ in (57)-(58) yields

$$\begin{aligned} &\{(1+\alpha^2)(5\alpha\lambda_1 + \beta) - 4\alpha^2\beta\}\{6\beta\lambda_1 - (2\lambda_1^2 + S)\alpha\} \\ &+ (22\alpha^2\lambda_1^2 - 12\alpha\beta\lambda_1 + 2\lambda_1^2 + S - \alpha^2 S)\{\lambda_1(\alpha^2+1) + \alpha\beta\} \\ &+ 6\beta(2\alpha\beta - 3\alpha^2\lambda_1 - \lambda_1)(\alpha\lambda_1 + \beta) + 3(1+\alpha^2)\lambda_1 S = 0. \end{aligned} \tag{59}$$

Applying (50) to eliminate the terms of S in (59), we derive

$$\begin{aligned}&8\alpha^6\lambda_1^3-20\alpha^5\beta\lambda_1^2+16\alpha^4\beta^2\lambda_1-4\alpha^3\beta^3\\&+9\alpha^4\lambda_1^3-14\alpha^3\beta\lambda_1^2+8\alpha^2\beta^2\lambda_1-2\alpha\beta^3\\&-6\alpha^2\lambda_1^3+6\alpha\beta\lambda_1^2-4\beta^2\lambda_1-7\lambda_1^3=0.\end{aligned}\tag{60}$$

Differentiating (60) along e_1 and using (46), (47) and (53), we have

$$\begin{aligned}&24\alpha^9\lambda_1^4-116\alpha^8\beta\lambda_1^3+188\alpha^7\beta^2\lambda_1^2-124\alpha^6\beta^3\lambda_1\\&+36\alpha^7\lambda_1^4-145\alpha^6\beta\lambda_1^3+191\alpha^5\beta^2\lambda_1^2-110\alpha^4\beta^3\lambda_1\\&-36\alpha^5\lambda_1^4+76\alpha^4\beta\lambda_1^3-48\alpha^3\beta^2\lambda_1^2+15\alpha^2\beta^3\lambda_1\\&-84\alpha^3\lambda_1^4+123\alpha^2\beta\lambda_1^3-51\alpha\beta^2\lambda_1^2+9\beta^3\lambda_1\\&-36\alpha\lambda_1^4+18\beta\lambda_1^3+28\alpha^5\beta^4+24\alpha^3\beta^4=0.\end{aligned}\tag{61}$$

Combining equations (60) and (61), we may eliminate β and obtain

$$\left\{484844765184\alpha^{65} + \cdots + 846526464\alpha^{7}\right\}\lambda_1^{16} = 0.$$

Since $\lambda_1 \neq 0$, the above equation is a non-trivial polynomial equation concerning α with constant coefficients. This implies that α must be a constant. It follows from (46) that

$$\beta = -\frac{\alpha^2+1}{\alpha}\lambda_1. \tag{62}$$

Substituting (62) into (47) and (53), we have

$$e_1(\lambda_1) = \lambda_1\Big(\alpha\lambda_1 - \frac{1+\alpha^2}{\alpha}\lambda_1\Big) = -\frac{1}{\alpha}\lambda_1^2, \tag{63}$$

$$\begin{aligned} e_1(\lambda_1) &= -\frac{4\alpha^3 + 6\alpha(\alpha^2+1) + \frac{2(1+\alpha^2)^2}{\alpha} + 4\alpha + \frac{2(1+\alpha^2)}{\alpha}}{1+\alpha^2}\lambda_1^2 \\ &= -\frac{4(3\alpha^2+1)}{\alpha}\lambda_1^2. \end{aligned} \tag{64}$$

Combining (63) with (64) yields that $4\alpha^2 + 1 = 0$, contradiction.

Thank You !