Producing Ricci flows by singular Ricci flows

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Structure of Talk:

- Part I Introduction
- Part II Singular Ricci flow
- Part III Generalized singular Ricci flow
- Part IV Proof of the main theorem

Part I Introduction

Introduction

Ricci flow equation:

$$\frac{d}{dt}g(t) = -2\mathrm{Ric}(g(t)) \tag{0.1}$$

Theorem (Hamilton)

Let M be a compact n-dimensional manifold, there exists a short time Ricci flow starting from M.

Compact RF preserves $\mathrm{Ric} \geq 0$ in 3d. Curvature blows up in finite time.

Theorem (Shi)

Let M be a complete n-dimensional manifold with bounded curvature, there exists a short time complete Ricci flow starting from M.

Shi's RF preserves $\mathrm{Ric} \geq 0$ in 3d.

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Theorem (Simon, Topping, 2017)

Let (M, g) be a 3d complete manifold. Suppose $Vol_g(B_g(x,1)) \geq v_0$ (non-collapsing) and Ric ≥ -1 everywhere (curvature lower bound). Then there exists a Ricci flow $(M, g(t))$, $t \in [0, T]$ with $g(0) = g$.

Idea: Take an exhaustion of M by compact subsets U_i. For each U_i, construct a local Ricci flow $(U_i, g_i(t))$, $t \in [0, T]$, by running Shi's Ricci flow inductively. Take a limit of $(U_i, g_i(t))$ to get a Ricci flow $(M, g(t))$.

Two key curvature estimates:

- $|\mathrm{Rm}|_{g_i(t)} \leq \frac{C}{t}$. Suppose this is not true, there is a sequence of Ricci flows converging to a κ-solution. The non-collapsing assumption implies the asymptotic volume ratio is non-zero, contradiction.
- Ric $\geq -C$, obtained by a bootstrap argument.

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Some invariant curvature conditions:

(1) non-negative curvature operator;
(2) non-negative complex sectional curvature (weakly PIC_2);
(3) 2-non-negative curvature operator ($\mathrm{Ric} \geq 0$ in 3d);
(4) weakly PIC_1;

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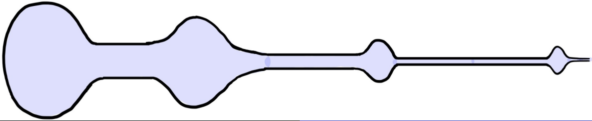
Remark: The non-collapsing assumption cannot be removed, if we only assume a negative lower bound on curvature:



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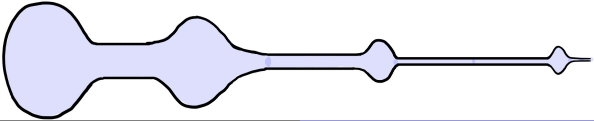

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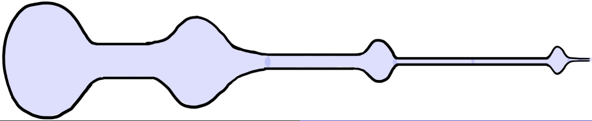

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In arbitrary dimension, there exists a complete Ricci flow starting from a complete manifold with non-negative complex sectional curvature.

Idea: take $(M_i, p_i) \to (M, p)$, where M_i is compact and has non-negative complex sectional curvature. So the same holds for $(M_i, g_i(t))$, $t \in [0, T_i]$, and $\lim_{t \nearrow T_i} Vol_t(M_i) = 0$. By Petrunin's result, $\int_{B_t(p_i,1)} R\, dvol \leq C$, it implies $T_i \geq T$ for all i. Then take a convergent subsequence of $(M_i, g_i(t))$, $t \in [0, T]$.

Note, in 3d, complex sec $\geq 0 \Leftrightarrow$ sec $\geq 0 \Rightarrow$ Ric ≥ 0.

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A conjecture by Topping

Given a 3d complete Riemannian manifold (M, g) with $\text{Ric} \geq 0$, there is a smooth continuation by Ricci flow.

Main theorem (L, 2020)

Given a 3d complete Riemannian manifold (M, g) with $\text{Ric} \geq 0$, there is a smooth Ricci flow $(M, g(t))$, $t \in [0, T_{max})$, starting out from (M, g). Moreover, if $T_{max} < \infty$, then the curvature blows up everywhere when t goes up to T_{max}.

e.g. $T_{max} < \infty$: the standard solution, $S^2 \times \mathbb{R}$; $T_{max} = \infty$: Bryant soliton

Strategy to construct the flow:

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Part II Singular Ricci flow

Singular Ricci flow

A Ricci flow spacetime $(\mathcal{M}, g(t))$ is the following:

- $\mathcal{M}$ is a 4-manifold with boundary.
- time function $\mathfrak{t} : \mathcal{M} \to [0, T)$, time-t-slice $\mathcal{M}_t$, and $\mathcal{M}_0 = \partial\mathcal{M}$.
- ∂_t is a smooth vector field in $\mathcal{M}$, $\partial_t \mathfrak{t} = 1$.
- g is a metric on $\ker(d\mathfrak{t})$.
- $\mathcal{L}_{\partial_t} g = -2\mathrm{Ric}(g(t))$.

Canonical neighborhood assumption (CNA): Let M be a 3d manifold. We say that the ϵ-CNA holds at $x \in M$, if (M, x) is ϵ-close to a κ-solution at scale $R(x)^{-1/2}$.

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ϵ-neck: A region that is ϵ-close to $S^2 \times \mathbb{R}$ under rescaling.

strong ϵ-neck: A spacetime region that is ϵ-close to $(S^2 \times \mathbb{R}, g(t))$ for $t \in [-1, 0]$ under rescaling.

gradient estimates: If ϵ-CNA holds at x, then

$$|\nabla R^{-1/2}|(x) \leq C, \quad |\partial_t R^{-1}|(x) \leq C. \tag{0.2}$$

We say a Ricci flow spacetime $\mathcal{M}$ is 0-complete (resp. backward 0-complete) if for any smooth curve $\gamma : [0, s_0) \to \mathcal{M}$ that satisfies $\inf_{[0,s_0)} R(\gamma(s)) < \infty$ and one of the following, then $\lim_{s \to s_0} \gamma(s)$ exists:

- $\gamma([0, s_0))$ is contained in a time-slice $\mathcal{M}_t$, and has finite length with respect to the horizontal metric in $\mathcal{M}_t$, or
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Theorem (Kleiner,Lott, 2014)

Let (M, g) be a 3d compact manifold, then there exists a **singular Ricci flow** starting from M, which is a Ricci flow spacetime that satisfies

- $\mathcal{M}_0 = M$ is compact;
- $\mathcal{M}$ is 0-complete;
- For any $x \in \mathcal{M}$, $\mathfrak{t}(x) \leq T$, if $R(x) \geq r^{-2}(T)$, then the ϵ-CNA holds at x.

Theorem: For any $x_0 \in M$, suppose x_0 survives until $t_0 > 0$, then

$$\mathcal{N} := \bigcup_{t=[0,t_0]} \bigcup_{A>0} B_t(x_0(t), A) \tag{0.3}$$

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Let $(\mathcal{M}, g(t))$ be a singular Ricci flow with normalized initial condition, $x_0 \in \mathcal{M}$, $\mathfrak{t}(x_0) = t_0$. Suppose $|\mathrm{Rm}| \leq r_0^{-2}$ in $\mathcal{P}_0 := P(x_0, t_0, r_0, -r_0^2)$, then

Theorem (Heat kernel)

Then there is a solution $u \geq 0$ to $(-\partial_t - \Delta + R)u = 0$, u is a δ-function at x_0, and $C_m = C_m(r_0)$, such that

$$uR^m \leq C_m \quad \text{in} \quad \mathcal{M}_{t<t_0} - \mathcal{P}_0 \tag{0.4}$$

Step 1 (construct u): Let $\mathcal{M}_i \to \mathcal{M}$ be a sequence of Ricci flow with surgeries. Define u_i on $\mathcal{M}_i$ by integrating with the ordinary heat kernels. Then $u_i \to u$.

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Step 2 (a vanishing theorem): Studying the solution $u \geq 0$ to $(-\partial_t - \Delta + R)u = 0$ in a non-compact κ-solution on $[0, T_{max})$.

For example, in a Bryant soliton: If $uR^m \leq C$, then $u \equiv 0$.

Step 3 (a semi-local maximum principle): For any x_1 with sufficiently large R, there is x_2 with $\mathfrak{t}(x_2) \geq \mathfrak{t}(x_1)$ such that

$$\begin{cases} uR^m(x_2) \geq (1+\epsilon_m)uR^m(x_1), \\ u(x_2) \geq (1+\epsilon_m)u(x_1). \end{cases} \tag{0.5}$$

Prove (0.5) by a limiting argument: Suppose it is violated in a sequence $(\mathcal{M}_i, x_i, u_i)$, with $R(x_i) \to \infty$. Then rescale each flow by $R(x_i)$, and rescale u_i such that $u_i(x_i) = 1$. Then

$$(\mathcal{M}_i, x_i, u_i) \to (g_\infty(t), x_\infty, u_\infty), \tag{0.6}$$

where $g_\infty(t)$ is a non-compact κ-solution defined on $[0, T_{\max})$. By step 2 we get a contradiction. Prove the theorem by using (0.5) repeatedly.

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Singular Ricci flow

Corollary: $\int_{\mathcal{M}_t} u \, d_t vol = 1$ for all $t \in [0, t_0)$.

Corollary: Pseudolocality theorem for singular Ricci flows.

Canonical neighborhood theorem

Let $(\mathcal{M}, g, x_0)$ be a singular Ricci flow, $x_0 \in \mathcal{M}_0$. Suppose $|\mathrm{Rm}| \leq 1$ and $vol(B_1(x_0, 1)) \geq A^{-1}$ on $P(x_0, 0; 1, 1)$. Then there exists $r(A) > 0$ such that the ϵ-CNA holds in $B_1(x_0, A)$ at scales less than $r(A)$.

Remark: Unlike the case of Ricci flow with surgeries, there is no need to assume that the initial condition of $\mathcal{M}$ is normalized, thanks to the 'zero-surgery scale' of the singular Ricci flow.

Singular Ricci flow

Corollary: $\int_{\mathcal{M}_t} u\, d_t vol = 1$ for all $t \in [0, t_0)$.

Corollary: Pseudolocality theorem for singular Ricci flows.

Canonical neighborhood theorem

Let $(\mathcal{M}, g, x_0)$ be a singular Ricci flow, $x_0 \in \mathcal{M}_0$. Suppose $|\mathrm{Rm}| \leq 1$ and $vol(B_1(x_0, 1)) \geq A^{-1}$ on $P(x_0, 0; 1, 1)$. Then there exists $r(A) > 0$ such that the ϵ-CNA holds in $B_1(x_0, A)$ at scales less than $r(A)$.

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Part III Generalized singular Ricci flow

Generalized singular Ricci flow

Theorem (L, 2020)

Let (M, g) be a 3d complete manifold (with possibly unbounded curvature). Then there exists a **generalized singular Ricci flow** $\mathcal{M}$ starting from (M, g), which is a Ricci flow spacetime that satisfies:

- $\mathcal{M}_0 = M$ is complete;
- $\mathcal{M}$ is 0-complete;
- For any fixed $x_0 \in \mathcal{M}$, $\mathfrak{t}(x_0) = t_0$, ϵ-CNA holds on $B_{t_0}(x_0, A)$ at scales $(0, r(A))$.

Generalized singular Ricci flow

Proof of the Theorem:

Pick $x_0 \in M$, and a sequence of compact manifolds

$$(M_i, x_{0i}) \longrightarrow (M, x_0). \tag{0.7}$$

Let $\mathcal{M}_i$ be singular Ricci flows with $\mathcal{M}_{i,0} = M_i$.

By the pseudolocality theorem,

$x \in B_t(x_{0i}, A)$, $t \in [0, t(A)] \Rightarrow |\mathrm{Rm}|(x) \leq C(A)$.

Take $T = t(10)$. By the canonical neighborhood theorem,

$x \in B_t(x_{0i}, A)$, $t \in [t(A), T] \Rightarrow \epsilon$-CNA holds if $|\mathrm{Rm}| \geq r(A)^{-2}$.

In summary, by decreasing $r(A)$, we have

$x \in B_t(x_{0i}, A)$, $t \in [0, T] \Rightarrow \epsilon$-CNA holds if $|\mathrm{Rm}| \geq r(A)^{-2}$.

Therefore, for any fixed A, $B_t(x_{0i}, A)$ is uniformly totally bounded.

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Generalized singular Ricci flow

Let $G_i = dt^2 + g_i(t)$, and d_i be the metric induced by G_i.

Let $P_i(A) := \bigcup_{t \in [0,T)} B_t(x_{0i}, A)$. Then $(P_i(A), d_i)$ is uniformly totally bounded. So

$$(P_i(A), d_i) \xrightarrow{GH} (X(A), d_A). \tag{0.8}$$

Let $\mathcal{N}_i = \bigcup_{A>0} P_i(A)$, then

$$(\mathcal{N}_i, d_i, x_{0i}) \xrightarrow{pGH} (X, d, x_0). \tag{0.9}$$

Let $\mathcal{M} = \{$'smooth points' in $X\}$. By the gradient estimate, there is a smooth spacetime metric on $\mathcal{M}$, $\mathfrak{t}(M) = [0, T)$, and

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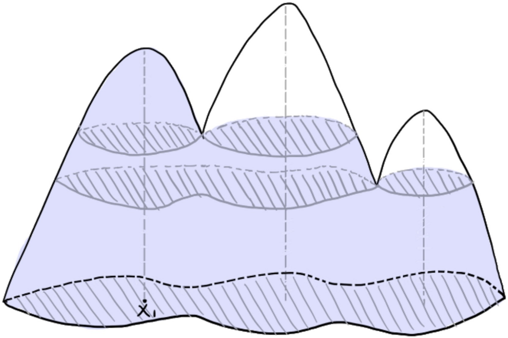
Generalized singular Ricci flow

By taking T maximal, we can assume that x_0 survives until its curvature blows up.

Moreover, we can show that $(\mathcal{M}, x_0)$ is backward 0-complete. By taking a 'union' of all such $(\mathcal{M}, x_0)$ we get a generalized singular Ricci flow.



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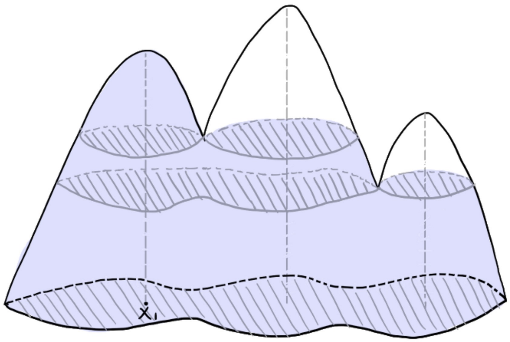

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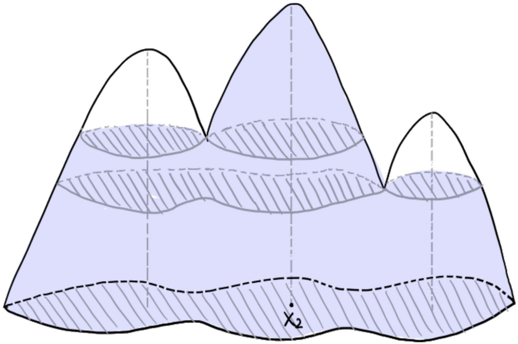

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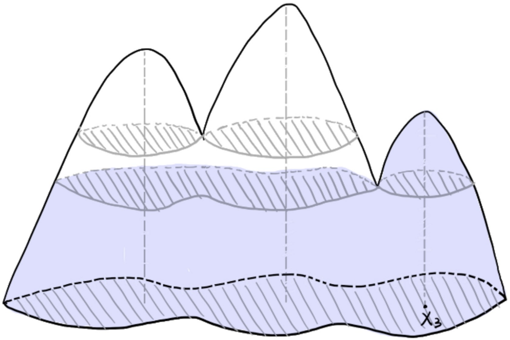

Part IV Proof of the main theorem

Proof of the main theorem

Lemma

Let (M, g) be a complete 3-manifold with $\mathrm{Ric} \geq 0$ (resp. $R \geq 0$). Let $\mathcal{M}$ be a generalized singular Ricci flow starting from (M, g). Then $\mathrm{Ric} \geq 0$ (resp. $R \geq 0$) on $\mathcal{M}$.

To show $R \geq 0$ is preserved, note

- In each $\mathcal{M}_t$, R is positive in the high curvature regions. So $R_{\min} < 0$ is achieved at some point.
- $\bigcup_{t\in[0,T)} \bigcup_{A>0} B_t(x_0(t), A)$ is backward 0-complete. It guarantees

$$\liminf_{t \searrow t_0} R_{\min}(t) \geq R_{\min}(t_0). \tag{0.11}$$

Then apply maximum principle.

We can show $\mathrm{Ric} \geq 0$ in a similar way.

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Proof of the main theorem

Main theorem (L, 2020)

Given a 3d complete Riemannian manifold (M, g) with $\mathrm{Ric} \geq 0$, there is a smooth Ricci flow $(M, g(t))$, $t \in [0, T_{max})$, starting out from (M, g). Moreover, if $T_{max} < \infty$, then the curvature blows up everywhere when t goes up to T.

Proof: Let $(\mathcal{M}, g(t))$ be a generalized singular Ricci flow starting from M. Let $x_0 \in M$. Suppose x_0 survives until $T > 0$. We claim that $\mathcal{M}_t$ is complete for all $t \in [0, T]$.

Suppose not, then for some $t, A > 0$ there is a minimizing geodesic $\gamma : [0, 1) \to B_t(x_0, A)$ such that $\lim_{s \to 1} R(\gamma(s)) = \infty$, and $\gamma(s)$ is center of strong ϵ-necks for all s close to 1.

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Proof of the main theorem

Main theorem (L, 2020)

Given a 3d complete Riemannian manifold (M, g) with Ric ≥ 0, there is a smooth Ricci flow $(M, g(t))$, $t \in [0, T_{max})$, starting out from (M, g). Moreover, if $T_{max} < \infty$, then the curvature blows up everywhere when t goes up to T.

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Proof of the main theorem

Let $X = \{p\} \cup B_t(x_0, A)$ be the one-point completion, and take a blow-up limit of X at p,

$$\lambda X \xrightarrow{GH} X_\infty, \text{ as } \lambda \to \infty. \tag{0.12}$$

Then by Ric ≥ 0, we can show X_∞ is a smooth cone.

Since for any $x \in X_\infty$, x is the center of a strong 2ϵ-neck, X_∞ is flat.

However, by the gradient estimate on X,

$$|\nabla R^{-\frac{1}{2}}| \leq C \Rightarrow R^{-\frac{1}{2}}(x) \leq C\, d(x, p). \tag{0.13}$$

So X_∞ is not flat, a contradiction. So $\mathcal{M}_t$ is complete for all $t \in [0, T]$.

Since Ric ≥ 0, we have $d_t(x, x_0) \leq d_0(x, x_0)$ for any $x \in M$. So M survives until T, and $M \times [0, T] \subset \mathcal{M}$ is a smooth Ricci flow.

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Thanks for your listening!