Dynamics of nonlinear wave equations

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Chapter 1

Introduction

1.1 Deriving the linear wave equation from a spring system

1.1.1 Using Newton's second law

Recall how the wave equation is derived in elementary dynamics. Let $h = \frac{1}{N+1} > 0$ and suppose that for each $n \in \{0, 1, \ldots, N+1\}$ we have a node at the point $x_n := nh \in [0, 1] \in \mathbb{R}$, connected with its neighbours with springs of length $lh < h$. The nodes number 0 and $N+1$ are fixed, whereas all the others can be moved in the vertical direction. We denote the vertical direction u and we call $u_n = u_n(t)$ the vertical displacement of the n-th node at time t.

Let $n \in \{1, \ldots, N\}$. We compute the vertical component of the force acting on the n-th node. Denote $\Delta u_l := u_{n-1} - u_n$, $\Delta u_r := u_{n+1} - u_n$, $l_l := \sqrt{h^2 + (\Delta u_l)^2}$ and $l_r := \sqrt{h^2 + (\Delta u_r)^2}$. We have

$$F = F_l + F_r = \frac{\Delta u_l}{l_l} \frac{k}{h}(l_l - lh) + \frac{\Delta u_r}{l_r} \frac{k}{h}(l_r - lh)$$

If we assume

$$|\Delta u_l| \ll h, \qquad |\Delta u_r| \ll h,$$

then

$$1 - \frac{lh}{l_l} \simeq 1 - l, \qquad 1 - \frac{lh}{l_r} \simeq 1 - l,$$

so we obtain

$$F \simeq k(1 - l/h)(\Delta u_l + \Delta u_r) = k(1 - l)(u_{n-1}(t) + u_{n+1}(t) - 2u_n(t)).$$

If we assume that all the nodes have the same mass mh, then the Newton's 2nd law leads to the system:

$$u_1''(t) = \frac{k_0}{h^2}(u_2(t) - 2u_1(t)), \qquad u_N''(t) = \frac{k_0}{h^2}(u_{N-1}(t) - 2u_N(t)),$$
$$u_n''(t) = \frac{k_0}{h^2}(u_{n-1}(t) + u_{n+1}(t) - 2u_n(t)), \qquad \text{for all } n \in \{1, \ldots, N\},$$

with $k_0 := \frac{(1-l)k}{m}$.

Now suppose we take $h \to 0$, so that we consider a continuous medium. Let $u(t,x)$ be the vertical displacement at time t, for $x \in [0,1]$. Then $u_n(t) = u(t,nh)$, and $\frac{1}{h^2}(u_{n-1}(t) + u_{n+1}(t) - 2u_n(t)) \simeq \partial_x^2 u(t,x)$. We obtain the *wave equation on the interval* $[0,1]$ *with Dirichlet boundary conditions*:

$$u(t,0) = u(t,1) = 0,$$
$$\partial_t^2 u(t,x) = k_0 \partial_x^2 u(t,x).$$

It is also called the *equation of a vibrating string.*

Analogously, we could consider displacements at any point $x \in \Omega \subset \mathbb{R}^d$, with values in $\mathbb{R}^m$, obtaining

$$\begin{aligned} u(t,x) &= 0, \qquad \text{for all } x \in \partial\Omega, \\ \partial_t^2 u^j(t,x) &= k_0 \Delta u^j(t,x), \qquad j \in \{1,\dots,m\}, \end{aligned} \tag{1.1.1}$$

where $\partial\Omega$ is the boundary of Ω and $\Delta := \partial_{x_1}^2 + \dots \partial_{x_d}^2$ is the Laplace operator in $\mathbb{R}^d$. For $d = 2$ and $m = 1$, it describes a vibrating membrane (like in a drum).

1.1.2 Using Lagrange variational principle

We consider directly the general case $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^m$. Each node $nh \in \Omega$ has mass mh^d. The kinetic energy is given by

$$T = \sum_{nh \in \Omega} \frac{mh^d}{2} |u_n'(t)|^2.$$

The potential energy is given (at first order) by

$$V = \sum_{|\tilde{n}-n|=1} \frac{kh^{d-2}}{2} (\sqrt{h^2 + |u_{\tilde{n}}(t) - u_n(t)|^2} - lh)^2 \simeq \sum_{|\tilde{n}-n|=1} \frac{(1-l)kh^d}{2} \Big((1-l) + \frac{|u_{\tilde{n}}(t) - u_n(t)|^2}{h^2}\Big).$$

The Lagrangian is $L = T - V$, and the Euler-Lagrange equations read

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial u_n'}\Big) = \frac{\partial L}{\partial u_n} \;\Leftrightarrow\; u_n''(t) = \frac{k_0}{h^2}\Big(\sum_{|\tilde{n}-n|=1} u_{\tilde{n}}(t) - 2du_n(t)\Big),$$

which are the Newton equations which we found before. Instead of passing to $h \to 0$ in the equations, we can pass to the limit in the Lagrangian and then use the variational principle.

Passing to $h \to 0$, we have

$$T \to \frac{m}{2} \int_\Omega |\partial_t u(t,x)|^2 \,\mathrm{d}x, \qquad V \to \frac{(1-l)k}{2} \int_\Omega |\nabla_x u(t,x)|^2 \,\mathrm{d}x,$$

so that

$$\frac{1}{m} L \to \int_\Omega \Big(\frac{1}{2}|\partial_t u(t,x)|^2 - \frac{k_0}{2}|\nabla_x u(t,x)|^2\Big) \,\mathrm{d}x$$

We have $\frac{\partial L}{\partial u} = \Delta u$, where Δ is the Dirichlet Laplacian in Ω. The Euler-Lagrange equation is

$$\partial_t^2 u = k_0 \Delta u.$$

In the sequel, we usually choose units so that $m = k_0 = 1$. Note that, if we consider a compact Riemannian manifold $(\mathcal{M}, g)$ instead of Ω, the Laplacian is replaced by the Laplace-Beltrami operator (which is defined as $-\frac{\partial}{\partial u} \int_{\mathcal{M}} \frac{1}{2} |\nabla u|_g^2 \,\mathrm{d}x$).

1.2 Nonlinearities

Equation (1.1.1) is linear. A nonlinearity can appear in at least three ways:

1. The interaction (potential) energy is not quadratic.
2. We apply an external potential which is not quadratic.
3. The nodes are constrained to lie on a sub-manifold of $\mathbb{R}^m$.

We will not consider the first option here.

1.2.1 Adding a potential

Suppose that the nodes are subject to an external potential $W(x,u)$. Then the potential energy becomes

$$V = \int_\Omega \Big(\frac{1}{2}|\nabla_x u(t,x)|^2 + W(x,u)\Big)\,\mathrm{d}x,$$

and the corresponding wave equation is

$$\partial_t^2 u(t,x) = \Delta u(t,x) - \partial_u W(x, u(t,x)). \tag{1.2.1}$$

In a special case $W(x,u) = q(x)u^2$, the equation is still linear: $\partial_t^2 u = (\Delta - q(x))u$. If $q(x) = \mathrm{const} > 0$, this is the linear Klein-Gordon equation. However, in some well-known models the potential is not quadratic in u, and the resulting equation is nonlinear. For example, taking $d = m = 1$ and $W(x,u) = W(u) = \frac{1}{4}(1-u^2)^2$, we obtain the so-called ϕ^4 model:

$$\partial_t^2 u = \partial_x^2 u + u - u^3.$$

1.2.2 Constrained motion

Another typical situation leading to a nonlinear model happens when for some reasons $u(t,x)$ is required to belong to a manifold $\mathcal{N} \subset \mathbb{R}^m$. In this case the Euler-Lagrange equation is:

$$\partial_t^2 u - \Delta u \perp T_u\mathcal{N}, \qquad \text{for all } (t,x) \in \mathbb{R}\times\Omega. \tag{1.2.2}$$

In general, this is a nonlinear equation, called the *wave maps equation*.

The potential energy is given by

$$V = \int_\Omega \frac{1}{2}|\nabla_x u(t,x)|^2\,\mathrm{d}x,$$

and its critical points are the *harmonic maps* form Ω to $\mathcal{N}$.

1.3 Objectives of the course

In order to study mathematically models like (1.2.1) or (4.5.1), first we should study the *local well-posedness* or *Cauchy theory*. The aim is to give a precise definition of a solution and prove that for any initial data $(u(0,x), \partial_t u(0,x))$ belonging to a certain functional space, there exists an open time interval $0 \ni I$ and a unique solution $u(t,x)$. Moreover, the solution at a given time t should depend continuously on the initial data. The first part of the course will be devoted to the Cauchy theory.

We saw that the wave equation can be viewed as an analogue of the Newton equation

$$u''(t) = -\nabla V(u(t)), \qquad u(t) \in \mathbb{R}^N, \tag{1.3.1}$$

with an infinite-dimensional phase space. For equation (1.3.1), the problem of local well-posedness is settled using the Picard theorem. However, we are often interested in *dynamics of solutions*, that is what happens to solutions *after a long time*. Note that this question remains largely open in many seemingly simple cases, probably the most famous example being the problem of stability of the solar system.

Example 1.3.1. Some solutions of the Newton equation $u''(t) = u^2(t)$ *blow up*, which means cease to exist in finite time.

Example 1.3.2. Assume the following *growth condition* holds: $|\nabla V(u)| \leq C|u|$ for some $C > 0$. Then for any initial data $(u(0), u'(0)) = (u_0, \dot u_0) \in \mathbb{R}^{2N}$ the corresponding solution $u(t)$ *is global*, by which we mean it exists for all $t \in \mathbb{R}$.

Example 1.3.3. Assume 0 is a strict local minimum of V. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that if $|(u_0, \dot u_0)| \leq \delta$, then the corresponding solution $u(t)$ exists globally and $|(u(t), u'(t))| \leq \epsilon$ for all t.

In the last example, the energy $E(u(t), u'(t)) = \frac{1}{2}|u'(t)|^2 + V(u(t))$ plays a crucial role. In particular, in the situation of Example 1.3.3, we obtain that there exists $E_1 > 0$ such that if $E(u_0, \dot u_0) \leq E_1$ and u_0 belongs to the same connected component of $\{u : V(u) \leq E_1\}$ as 0, then the solution $u(t)$ exists globally. A natural question is to have some lower bound on E_0.

Definition 1.3.4. We say that E_c is a *critical value* of V if there exists a sequence $u_n \in \mathbb{R}^N$ such that $V(u_n) \to E_c$ and $\nabla V(u_n) \to 0$.

Proposition 1.3.5. *Assume that* 0 *is a strict local minimum of* V *and* E_0 *is such that* V *has no critical values in* $(0, E_0]$. *Then for any initial data* $(u_0, \dot u_0)$ *such that* $E(u_0, \dot u_0) \leq E_0$ *the corresponding solution* $u(t)$ *exists for all time and is bounded.*

Proof. It suffices to prove that the connected component of $\{u : V(u) \leq E_0\}$ containing 0 is bounded. By assumption, there exists $E_1 > 0$ such that the connected component of $\{u : V(u) \leq E_1\}$ containing 0 is bounded. Let $\Phi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a modified gradient flow of V:

$$\partial_t \Phi(t, u) = -\frac{\nabla V(u)}{1 + |\nabla V(u)|^2}.$$

By assumption, there exists $\epsilon \in (0, 1)$ such that $E_1 \leq V(u) \leq E_0$ implies $|\nabla V(u)| \geq \epsilon$. Let $T := \frac{2(E_0 - E_1)}{\epsilon^2}$. We will check that

$$\Phi(T, \{u : V(u) \leq E_0\}) \subset \{u : V(u) \leq E_1\}.$$

Let $u_0 \in \{u : V(u) \leq E_0\}$, $u(t) := \Phi(t, u_0)$ and suppose that $u(T) > E_1$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} V(u(t)) = \nabla V(u(t)) \cdot \Big(-\frac{\nabla V(u(t))}{1 + |\nabla V(u(t))|^2}\Big) = -\frac{|\nabla V(u(t))|^2}{1 + |\nabla V(u(t))|^2} \leq -\frac{\epsilon^2}{1 + \epsilon^2} \leq -\frac{\epsilon^2}{2},$$

thus

$$u(T) \leq u(0) - \frac{T\epsilon^2}{2} \leq E_0 - (E_0 - E_1) = E_1,$$

a contradiction.

Since the image of a connected set by a continuous mapping is connected, we obtain that the image of the connected component of $\{u : V(u) \leq E_0\}$ containing 0 is contained in the connected component of $\{u : V(u) \leq E_1\}$ containing 0, thus bounded. We also have

$$|u(0)| \leq |u(T)| + \int_0^T |u'(t)|\,\mathrm{d}t \leq T\frac{1}{2} = |u(T)| + \frac{E_0 - E_1}{\epsilon^2}.$$

□

As for the Newton equation, we can ask questions about the *dynamical behaviour* of solutions of wave equations. We will provide some sufficient conditions for $W(x, u)$ under which the solutions of (1.2.1) are global. We will study an analogous problem for (4.5.1). The variational structure of the potential energy, and in particular existence of critical energies, is going to play a crucial role.

1.4 Exercises

Exercise 1.4.1. Consider the differential equation from Example 1.3.1. Prove that if a solution $u(t)$ exists for all $t \geq 0$, then $u(t) = 6(t - t_0)^{-2}$ for some $t_0 > 0$. Similarly, if $u(t)$ exists for all $t \leq 0$, then $u(t) = 6(t_0 - t)^{-2}$ for some $t_0 > 0$. Thus "almost all" solutions blow up both for positive and negative times.
Hint: You might use the fact that the energy $\frac{1}{2}(u')^2 - \frac{1}{3}u^3$ is conserved and that u' is increasing. Perhaps one could also deduce this from the phase portrait.

Exercise 1.4.2. Let $u(t)$ be a solution of the equation in Example 1.3.2, defined on the maximal interval of existence (T_-, T_+). Prove that there exists $\widetilde{C} > 0$ such that $|(u(t), u'(t))| \leq \widetilde{C}\mathrm{e}^{\widetilde{C}|t|}$ for all $t \in (T_-, T_+)$. Deduce that $T_- = -\infty$ and $T_+ = +\infty$.

Exercise 1.4.3. Prove the statement in Example 1.3.3.

Chapter 2

Elements of harmonic analysis

2.1 Riesz-Thorin interpolation theorem

Lemma 2.1.1 (Three-line theorem, Phragmen-Lindelöf principle)**.** *Let $F(z)$ be bounded and continuous on the strip $0 \leq x \leq 1$ and analytic inside. If $|F(it, y)| \leq M_1$ and $F(1 + it, y) \leq M_2$ for all y, then*

$$|F(x, y)| \leq M_1^{1-x} M_2^x, \qquad \textit{for all } x \in [0, 1].$$

Proof. It is sufficient to consider $M_1 = M_2 = 1$. By considering the function $\widetilde{F}(z) := F(z)\mathrm{e}^{\epsilon(z^2-1)}$, we reduce to the case $\lim_{y\to\infty} |F(z)| = 0$, and the conclusion follows from the Maximum Principle. □

Proposition 2.1.2 (Riesz-Thorin interpolation theorem)**.** *Let (X, μ) and $(\widetilde{X}, \widetilde{\mu})$ be measure spaces. Let $1 \leq p_1, p_2 \leq \infty$ and assume that $Y \subset L^{p_1}(X, \mu) \cap L^{p_2}(X, \mu)$ is dense in both $L^{p_1}(X, \mu)$ and $L^{p_2}(X, \mu)$. Let T be a linear operator defined on Y taking its values in measurable functions on $(\widetilde{X}, \widetilde{\mu})$ and assume that $1 \leq q_1, q_2 \leq \infty$, M_1, M_2 are such that*

$$\|Tf\|_{L^{q_j}(\widetilde{X},\widetilde{\mu})} \leq M_j \|f\|_{L^{p_j}}(X, \mu), \qquad \textit{for all } f \in Y \textit{ and } j \in \{1, 2\}.$$

Then for all $\theta \in [0, 1]$

$$\|Tf\|_{L^q(\widetilde{X},\widetilde{\mu})} \leq M_1^\theta M_2^{1-\theta} \|f\|_{L^p(X,\mu)} \qquad \textit{for all } f \in Y,$$

where

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Proof. The conclusion is obvious if $\theta = 0$ or $\theta = 1$, so assume $0 < \theta < 1$. If $p_1 = p_2 = \infty$, then the theorem follows from the Hölder inequality, thus we may assume $p_1 < \infty$ or $p_2 < \infty$, which allows us to consider only f being a step function with finite set of values. Note that we can assume that Y contains such functions (extending T by density if needed; we could also assume that $Y = L^{p_1} \cap L^{p_2}$).

We need to estimate

$$\sup\{\langle Tf, g\rangle : \|f\|_{L^p} \leq 1, \|g\|_{L^{q'}} \leq 1\},$$

with the supremum taken over step functions with a finite set of values:

$$f = \sum_j a_j I_{A_j}, \qquad g = \sum_k b_k I_{B_k}.$$

(Attention to the case $q = q_1 = q_2 = 1$).

For $0 \le \Re z \le 1$ we set

$$\frac{1}{p(z)} := \frac{1-z}{p_1} + \frac{z}{p_2}, \qquad \frac{1}{q'(z)} := \frac{1-z}{q_1'} + \frac{z}{q_2},$$

$$\phi(z) := \sum_j |a_j|^{\frac{p}{p(z)}} \mathrm{e}^{i \arg a_j} I_{A_j}, \quad \psi(z) := \sum_k |b_k|^{\frac{q'}{q'(z)}} \mathrm{e}^{i \arg b_k} I_{B_k}.$$

We apply the three-line theorem to $\langle T\phi(z), \psi(z)\rangle$. □

2.2 Real analysis

In this section, we follow Chapter 1 from the book [1].

Proposition 2.2.1 (Minkowski inequality)**.** *If (X, μ), (Y, ν) measure spaces, $1 \le p \le q \le \infty$ and $f : X \times Y \to \mathbb{R}_+$ is measurable, then*

$$\big\| y \mapsto \|f(\cdot, y)\|_{L^p(X)} \big\|_{L^q(Y)} \le \big\| x \mapsto \|f(x, \cdot)\|_{L^q(Y)} \big\|_{L^p(X)}.$$

Proof. We can assume that $f \ge 0$ and, upon replacing f by f^p, also that $p = 1$. Let $g \in L^{q'}(Y)$. We have

$$\int_Y g(y) \int_X f(x, y)\, \mathrm{d}x\, \mathrm{d}y \le \int_X \|f(x, \cdot)\|_{L^q} \|g\|_{L^{q'}}\, \mathrm{d}x$$

by Hölder inequality. □

2.2.1 Young inequalities for convolutions

Recall that for f, g functions on $\mathbb{R}^d$ we denote

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x-y) g(y)\, \mathrm{d}y,$$

whenever this expression makes sense.

Proposition 2.2.2 (Young's inequality)**.** *Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$. If*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$

then

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}.$$

Proof. If $q = 1$, this follows from Minkowski inequality. If $q = p'$ and $r = \infty$, this follows from Hölder inequality. The remaining cases follow from Riesz-Thorin. □

2.2.2 Weak L^p spaces

For a measurable function g we define

$$\|g\|_{L^q_w}^q := \sup_{\lambda>0} \lambda^q \mu\{x : |g(x)| \geq \lambda\}.$$

Lemma 2.2.3 (Markov inequality). *For any measurable g, $\|g\|_{L^q_w} \leq \|g\|_{L^q}$.* □

Proposition 2.2.4 (Refined Young's inequality). *Under assumptions of Proposition 2.2.2, if $1 < p, q, r < \infty$, there exists $C > 0$ such that for all measurable f, g*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q_w}.$$

Corollary 2.2.5 (Hardy-Littlewood-Sobolev inequality). *If $\alpha \in (0, d)$ and $(p, r) \in (1, \infty)$ satisfy*

$$\frac{1}{p} + \frac{\alpha}{d} = 1 + \frac{1}{r},$$

then

$$\||\cdot|^{-\alpha} * f\|_{L^r} \leq C\|f\|_{L^p}.$$

Proof. The function $|x|^{-\alpha}$ is in the space $L^{d/\alpha}_w(\mathbb{R}^d)$. □

In order to prove the refined Young inequality, we use the following tool.

Proposition 2.2.6 (Atomic decomposition). *Let (X, μ) a measure space, $p \in [1, \infty)$, $f \in L^p(X)$ positive. There exist sequences of positive real numbers $(c_k)_{k\in\mathbb{Z}}$ and functions $(f_k)_{k\in\mathbb{Z}}$ such that*

$$\operatorname{supp} f_j \cap \operatorname{supp} f_k = \emptyset,$$
$$\mu(\operatorname{supp} f_k) \leq 2^{k+1},$$
$$\|f_k\|_{L^\infty} \leq 2^{-\frac{k}{p}},$$
$$\frac{1}{2}\|f\|_{L^p}^p \leq \sum_{k\in\mathbb{Z}} c_k^p \leq 2\|f\|_{L^p}^p.$$

Proof. We set

$$\lambda_k := \inf\{\lambda : \mu(f > \lambda) < 2^k\},$$
$$c_k := 2^{\frac{k}{p}} \lambda_k,$$
$$f_k := c_k^{-1} I_{\lambda_{k+1} < f \leq \lambda_k} f.$$

We will check all the requirements. □

Remark 2.2.7. Many other decompositions of this type are used in harmonic analysis. We will encounter at least one more example, the Littlewood-Paley decomposition.

Proof of Proposition 2.2.4. Next lecture. □

2.3 Fourier transform

In this section, the presentation is often close to the one in Chapter 4, Volume 1 of the book by Muscalu and Schlag [2].

Let μ be a complex-valued Borel measure on $\mathbb{R}^d$ of finite total variation. We define its Fourier transform:

$$(\mathcal{F}\mu)(\xi) = \widehat{\mu}(\xi) := \int_{\mathbb{R}^d} \mathrm{e}^{-ix\cdot\xi}\mu(\,\mathrm{d}x), \qquad \forall \xi \in \mathbb{R}^d.$$

We see that $\widehat{\mu}$ is a bounded continuous function.

If $f \in L^1(\,\mathrm{d}x)$, we set $\mathcal{F}f := \mathcal{F}(f\,\mathrm{d}x)$.

2.3.1 Fourier transform on the Schwartz space

It is useful to extend the Fourier transformation on functions which are not in L^1. In order to do this, we introduce the space of tempered distributions.

Definition 2.3.1. The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is the space of complex-valued functions $f \in C^\infty(\mathbb{R}^d)$ such that for any multi-indices $\alpha, \beta \in \mathbb{N}^d$

$$x^\alpha \partial^\beta f \in L^\infty(\mathbb{R}^d).$$

We say that a sequence $f_n \in \mathcal{S}(\mathbb{R}^d)$ converges to $f \in \mathbb{S}(\mathbb{R}^d)$ if for any multi-indices α, β

$$\lim_{n\to\infty} \|x^\alpha \partial^\beta (f_n - f)\|_{L^\infty} = 0.$$

Proposition 2.3.2. *The Fourier transform $\mathcal{F}$ is continuous $\mathcal{S} \to \mathcal{S}$.*

Proof. This follows from the formulas:

$$(i\partial)^\alpha \widehat{f}(\xi) = \mathcal{F}(x^\alpha f)(\xi),$$
$$(i\xi)^\alpha \widehat{f}(\xi) = \mathcal{F}(\partial^\alpha f)(\xi).$$

□

Proposition 2.3.3 (Fourier inversion theorem)**.** *The Fourier transform takes $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$. For any $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi} \widehat{f}(\xi)\,\mathrm{d}\xi, \qquad \forall x \in \mathbb{R}^d. \tag{2.3.1}$$

Proof. We need the following fact. For any $\epsilon > 0$ we have (see Exercise 2.6.2):

$$\int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi - \frac{\epsilon}{2}|\xi|^2}\,\mathrm{d}\xi = \left(\frac{2\pi}{\epsilon}\right)^{\frac{d}{2}} \mathrm{e}^{-\frac{|x|^2}{2\epsilon}}. \tag{2.3.2}$$

Using this, we can write, for any $\epsilon > 0$:

$$\begin{aligned}(2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi} \widehat{f}(\xi) \mathrm{e}^{-\frac{\epsilon}{2}|\xi|^2}\,\mathrm{d}\xi &= (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi} \int_{\mathbb{R}^d} \mathrm{e}^{-iy\cdot\xi} f(y) \mathrm{e}^{-\frac{\epsilon}{2}|\xi|^2}\,\mathrm{d}y\,\mathrm{d}\xi \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} \mathrm{e}^{i(x-y)\cdot\xi - \frac{\epsilon}{2}|\xi|^2}\,\mathrm{d}\xi\,\mathrm{d}y \\ &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(y) \epsilon^{-\frac{d}{2}} \mathrm{e}^{-\frac{1}{2}|\frac{x-y}{\sqrt{\epsilon}}|^2}\,\mathrm{d}y\end{aligned}$$

When $\epsilon \to 0$, the left hand side tends to the right hand side of (2.3.1), and the right hand side tends to $f(x)$. This finishes the proof. $\square$

Definition 2.3.4. A *tempered distribution* on $\mathbb{R}^d$ is a continuous linear functional on $\mathcal{S}(\mathbb{R}^d)$, that is a linear functional $\phi : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ such that $\langle \phi, u_n \rangle \to \langle \phi, u \rangle$ whenever $u_n \to u$ in $\mathcal{S}(\mathbb{R}^d)$.

We say that a sequence $\phi_n \in \mathcal{S}(\mathbb{R}^d)$ converges to $u \in \mathcal{S}(\mathbb{R}^d)$ if $\langle \phi_n, u \rangle \to \langle \phi, u \rangle$ for all $u \in \mathcal{S}(\mathbb{R}^d)$.

Proposition 2.3.5. *If $\phi \in \mathcal{S}'(\mathbb{R}^d)$, then there exists $C, N \geq 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^d)$*

$$|\langle \phi, u \rangle| \leq C \sum_{|\alpha| \leq N, |\beta| \leq N} \|x^\alpha \partial^\beta u\|_{L^\infty(\mathbb{R}^d)}.$$

Proof. Exercise 2.6.4. $\square$

Example 2.3.6. If f is locally integrable and there exists k such that $(1 + |x|)^{-k} f(x) \in L^1(\mathbb{R}^d)$, then we define $T_f \in \mathcal{S}'$ by the formula

$$\langle T_f, u \rangle = \int_{\mathbb{R}^d} f u \, \mathrm{d}x.$$

Note that traditionally we do not use the complex conjugate in this case.

Definition 2.3.7. For any continuous operator $A : \mathcal{S} \to \mathcal{S}$ we define the operator $A^t : \mathcal{S}' \to \mathcal{S}'$ by the formula:

$$\langle A^t \phi, u \rangle = \langle \phi, Au \rangle.$$

The Fubini theorem implies $\mathcal{F}^t u = \mathcal{F} u$ for $u \in L^1(\mathbb{R}^d)$, hence we will write $\mathcal{F}$ instead of $\mathcal{F}^t$. Analogously, we define $\partial^\alpha := (-1)^{|\alpha|} (\partial^\alpha)^t$ itp. If $\theta \in \mathcal{S}$, then we define $\phi * \theta$ by

$$\langle \phi * \theta, u \rangle = \Big\langle \phi, x \mapsto \int_{\mathbb{R}^d} \theta(y - x) u(y) \, \mathrm{d}y \Big\rangle.$$

Proposition 2.3.8. *The usual properties of the Fourier transform continue to hold. For any $u \in \mathcal{S}'$:*

$$\begin{aligned}
(i\partial)^\alpha \widehat{u} &= \mathcal{F}(x^\alpha u), \\
(i\xi)^\alpha \widehat{u} &= \mathcal{F}(\partial^\alpha u), \\
\mathrm{e}^{-ia\cdot\xi} \widehat{u}(\xi) &= \mathcal{F}(x \mapsto u(x - a))(\xi), \\
\widehat{u}(\xi - \omega) &= \mathcal{F}(\mathrm{e}^{ix\cdot\omega} u)(\xi), \\
\mathcal{F}(x \mapsto u(\lambda x))(\xi) &= \lambda^{-d} \widehat{u}(\xi/\lambda), \\
\mathcal{F}(u * \theta) &= \widehat{u}\widehat{\theta}, \qquad \textit{for all } \theta \in \mathcal{S}.
\end{aligned}$$

Proof. Exercise 2.6.5. $\square$

Proposition 2.3.9. *Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ be such that $\langle \phi, u \rangle$ for all $u \in \mathcal{S}(\mathbb{R}^d)$ with* $\operatorname{supp} u \subset \mathbb{R}^d \setminus \{0\}$. *Then $\widehat{\phi}$ is a polynomial, in other words ϕ is a finite linear combination of the Dirac delta and its derivatives.*

Proof. Exercise 2.6.6. $\square$

Proposition 2.3.10. *For any $\alpha \in (0,d)$ there exists $C(\alpha,d)$ such that*

$$\mathcal{F}(|x|^{-\alpha}) = C(\alpha,d)|\xi|^{\alpha-d}.$$

Remark 2.3.11. The functions $|x|^{-\alpha}$ are called *Riesz potentials.*

Proof. Exercise 2.6.7. □

Proposition 2.3.12 (Plancherel formula)**.** *For all $f \in L^2(\mathbb{R}^d)$,*

$$\|\mathcal{F}f\|_{L^2(\mathbb{R}^d)} = (2\pi)^{\frac{d}{2}}\|f\|_{L^2(\mathbb{R}^d)}.$$

Proof. Exercise 2.6.8. □

Proposition 2.3.13 (Hausdorff-Young inequality)**.** *For all $p \in [1,2]$ and $f \in L^p(\mathbb{R}^d)$ the inequality $\|\mathcal{F}f\|_{L^{p'}(\mathbb{R}^d)} \leq (2\pi)^{\frac{d}{p'}}\|f\|_{L^p(\mathbb{R}^d)}$ is true.*

Proof. This is clear for $p = 1$, for $p = 2$ follows from Proposition 2.3.12, and for the remaining values from the Riesz-Thorin theorem. □

Lemma 2.3.14 (Bernstein inequality)**.** *There exists $C_d \geq 0$ such that if $f \in \mathcal{S}(\mathbb{R}^d)$ is such that $\operatorname{supp}\widehat{f} \subset \{|\xi| \leq R\}$, then for any multi-index α*

$$\|\partial^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C(\alpha,d)R^{|\alpha|+d(1/p-1/q)}\|f\|_{L^p(\mathbb{R}^d)}, \qquad \textit{for all } 1 \leq p \leq q \leq \infty.$$

Proof. Considering $g(x) := f(x/R)$, we reduce the proof to the case $R = 1$. Indeed, $\widehat{g}$ is supported in the unit ball (see Proposition 2.3.8), $\|g\|_{L^p(\mathbb{R}^d)} = R^{d/p}\|f\|_{L^p(\mathbb{R}^d)}$ and $\|\partial^\alpha g\|_{L^q(\mathbb{R}^d)} = R^{-|\alpha|+d/q}\|\partial^\alpha f\|_{L^q(\mathbb{R}^d)}$.

In order to prove the lemma for $R = 1$, we write $\widehat{\partial^\alpha f}(\xi) = (i\xi)^\alpha\widehat{\chi}(\xi)\widehat{f}(\xi)$, where $\widehat{\chi} \in C^\infty$ is identically 1 on $\{|\xi| \leq 1\}$ and $\operatorname{supp}\chi \subset \{|\xi| \leq 2\}$. In particular $\chi \in \mathcal{S}(\mathbb{R}^d)$. Taking the inverse Fourier transform we obtain $\partial^\alpha f = f * \mathcal{F}^{-1}((i\xi)^\alpha\widehat{\chi})$. Let $r := (1/(p') + 1/q)^{-1} \geq 1$ (the last inequality follows from $q \geq p$). Proposition 2.2.2 yields

$$\|\partial^\alpha f\|_{L^q} \leq \|\mathcal{F}^{-1}((i\xi)^\alpha\widehat{\chi})\|_{L^r}\|f\|_{L^p} \leq C(\alpha,d)\|f\|_{L^p},$$

with

$$C(\alpha,d) := \max(\|\mathcal{F}^{-1}((i\xi)^\alpha\widehat{\chi})\|_{L^1}, \|\mathcal{F}^{-1}((i\xi)^\alpha\widehat{\chi})\|_{L^\infty}).$$

□

Remark 2.3.15. A more careful analysis shows that one can take $C(\alpha,d) = C_d^{1+|\alpha|}$, where C_d depends only on d.

2.4 Sobolev spaces

Definition 2.4.1. For any $s \in \mathbb{R}$ the *Sobolev space* H^s is defined as the completion of $\mathcal{S}$ in $\mathcal{S}'$ for the topology defined by the norm

$$\|f\|_{H^s} := \Big(\int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi \Big)^{\frac{1}{2}}.$$

Lemma 2.4.2. . *For any $s > \frac{d}{2}$ there is the inclusion $H^s(\mathbb{R}^d) \subset C(\mathbb{R}^d)$ and there exists $C_s \geq 0$ such that for all $f \in H^s(\mathbb{R}^d)$*

$$\|f\|_{L^\infty(\mathbb{R}^d)} \leq C_s \|f\|_{H^s(\mathbb{R}^d)}.$$

Proof. Exercise 2.6.9 □

We denote $\mathcal{S}_0$ the set of functions $u \in \mathcal{S}$ such that $\operatorname{supp} \widehat{u} \subset \mathbb{R}^d \setminus \{0\}$.

Definition 2.4.3. For any $s < \frac{d}{2}$ the *homogeneous Sobolev space* $\dot{H}^s$ is defined as the completion of $\mathcal{S}_0$ in $\mathcal{S}'$ for the topology defined by the norm

$$\|f\|_{\dot{H}^s} := \Big(\int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi \Big)^{\frac{1}{2}}.$$

Proposition 2.4.4 (Sobolev embedding). *Let $s < \frac{d}{2}$ and let $p > 0$ be determined by the relation*

$$\frac{1}{2} - \frac{1}{p} = \frac{s}{d} \;\Leftrightarrow\; p = \frac{2d}{d-2s}.$$

There exists a constant $C = C(s,d)$ such that

$$\|f\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{\dot{H}^s(\mathbb{R}^d)}, \qquad \text{for all } f \in \dot{H}^s(\mathbb{R}^d).$$

Proof. We can assume $f \in \mathcal{S}(\mathbb{R}^d)$ (for $f \in \dot{H}^s(\mathbb{R}^d)$ will follow by density). Let $g := \mathcal{F}^{-1}(|\xi|^s \widehat{f}(\xi))$, so that $f = \mathcal{F}^{-1}(|\xi|^{-s}) * g$. Now we use the Hardy-Littlewood-Sobolev inequality. □

2.4.1 Stationary and non-stationary phase

We now study *oscillatory integrals*, that is integrals of the form

$$I(\lambda) = \int_{\mathbb{R}^d} \mathrm{e}^{i\lambda\phi(\xi)} a(\xi) \,\mathrm{d}\xi,$$

where $a \in C_0^\infty(\mathbb{R}^d)$ and $\phi \in C^\infty(\mathbb{R}^d)$. We are interested in the asymptotic behaviour of $I(\lambda)$ as $\lambda \to +\infty$. Notice that if ϕ is a non-trivial affine function, then Proposition 2.3.2 implies that $|I(\lambda)|$ decays faster than any power of λ. The lemma below generalises this fact.

Lemma 2.4.5 (Non-stationary phase). *If $\nabla\phi \neq 0$ on $\operatorname{supp} a$, then for any $N \geq 1$ there exists $C(N,a,\phi) \geq 0$ such that*

$$|I(\lambda)| \leq C(N,a,\phi)\lambda^{-N}, \qquad \text{as } \lambda \to \infty.$$

Proof. Consider the differential operators

$$Lu := \frac{1}{i\lambda}\frac{\nabla\phi\cdot\nabla u}{|\nabla\phi|^2}, \qquad L^*u := \frac{i}{\lambda}\nabla\cdot\Big(\frac{u\nabla\phi}{|\nabla\phi|^2}\Big).$$

We have $L\mathrm{e}^{i\lambda\phi} = \mathrm{e}^{i\lambda\phi}$, hence integration by parts yields

$$|I(\lambda)| = \Big|\int_{\mathbb{R}^d} L^N \mathrm{e}^{i\lambda\phi(\xi)}a(\xi)\,\mathrm{d}\xi\Big| = \Big|\int_{\mathbb{R}^d} \mathrm{e}^{i\lambda\phi(\xi)}(L^*)^N a(\xi)\,\mathrm{d}\xi\Big| \le \int_{\mathbb{R}^d}\big|(L^*)^N a(\xi)\big|\,\mathrm{d}\xi \le C(N,a,\phi)\lambda^{-N}.$$

□

Lemma 2.4.6 (Stationary phase)**.** *Assume that all the critical points of ϕ belonging to* $\operatorname{supp} a$ *are non-degenerate, in other words*

$$\xi_0 \in \operatorname{supp} a \quad \textit{and} \quad \nabla\phi(\xi_0) = 0 \quad \Rightarrow \quad \det\big(\nabla^2\phi(\xi_0)\big) \neq 0.$$

Then there exists $C(a,\phi) \ge 0$ such that

$$|I(\lambda)| \le C(a,\phi)\lambda^{-\frac{d}{2}}, \qquad \textit{as } \lambda \to \infty.$$

Proof. Let $\chi \in C^\infty$ be a cut-off function, that is $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$. Since non-degenerate critical points are isolated, in $\operatorname{supp} a$ there is a finite number of them. Call them $\xi_1, \dots, \xi_m$. For each critical point ξ_j, let

$$I_j(\lambda) := \int_{\mathbb{R}^d} \mathrm{e}^{i\lambda\phi(\xi)}a(\xi)\chi\big(\sqrt{\lambda}(\xi-\xi_j)\big)\,\mathrm{d}\xi.$$

Obviously $|I_j(\lambda)| \le C(a)\lambda^{-\frac{d}{2}}$. Set

$$I_0(\lambda) := I(\lambda) - \sum_{j=1}^m I_j(\lambda) = \int_{\mathbb{R}^d} \mathrm{e}^{i\lambda\phi(\xi)}\widetilde{a}(\xi)\,\mathrm{d}\xi,$$

where $\widetilde{a}(\xi) := \big(1 - \sum_{j=1}^m \chi\big(\sqrt{\lambda}(\xi-\xi_j)\big)\big)a(\xi)$. From the non-degeneracy condition, there exists $c(a,\phi) > 0$ such that

$$|\nabla\phi(\xi)| \ge c(a,\phi)\sqrt{\lambda}, \qquad \forall \xi \in \operatorname{supp}\widetilde{a}.$$

□

We also have the following improved version.

Lemma 2.4.7. *Assume that ξ_0 is the only critical point of ϕ in* $\operatorname{supp} a$ *and that it is non-degenerate. Then for any $k \in \mathbb{N}$ there exists $C(k,a,\phi)$ such that*

$$\Big|\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\big(\mathrm{e}^{-i\lambda\phi(\xi_0)}I(\lambda)\big)\Big| \le C(k,a,\phi)\lambda^{-\frac{d}{2}-k}, \qquad \textit{as } \lambda \to \infty.$$

Proof. □

Corollary 2.4.8. *Let $\sigma_{S^{d-1}}(\xi)$ be the surface measure of the unit sphere $S^{d-1} \subset \mathbb{R}^d$. Then*

$$\mathcal{F}^{-1}\sigma_{S^{d-1}}(x) = \mathrm{e}^{i|x|}\omega_+(|x|) + \mathrm{e}^{-i|x|}\omega_-(|x|), \qquad |x| \ge 1,$$

where $\omega_\pm$ are smooth and for all $k \in \mathbb{N}$ there exists $C_k \ge 0$ such that

$$|\partial_r^k \omega_\pm| \le C_k r^{-\frac{d-1}{2}-k}, \qquad \textit{for all } r \ge 1.$$

Proof. □

2.5 Littlewood-Paley theory

Lemma 2.5.1 (Partition of unity over a geometric scale)**.** *There exists a radial nonnegative function* $\psi \in C^\infty(\mathbb{R}^d)$ *such that* $\operatorname{supp}\psi \subset \{\frac{1}{2} \le x \le 2\}$ *and*

$$\sum_{j=-\infty}^{\infty} \psi(2^{-j}x) = 1, \qquad \forall x \neq 0.$$

Proof. We take $\chi \in C^\infty$ a radial non-increasing cut-off function such that $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$. We set $\psi(x) := \chi(x) - \chi(2x)$. □

Definition 2.5.2. For $j \in \mathbb{Z}$ we define the *homogeneous dyadic block* $\dot\Delta_j$ and the *homogeneous low-frequency cut-off operator* $\dot S_j$:

$$\dot\Delta_j u := \psi(2^{-j}D)u := \mathcal{F}^{-1}(\psi(2^{-j}\xi)\widehat{u}(\xi)) = 2^{jd}\int_{\mathbb{R}^d}(\mathcal{F}^{-1}\psi)(2^j y)u(x-y)\,\mathrm{d}y,$$
$$\dot S_j u := \sum_{j'<j} \Delta_j u = \mathcal{F}^{-1}(\chi(2^{-j}\xi)\widehat{u}(\xi)) = 2^{jd}\int_{\mathbb{R}^d}(\mathcal{F}^{-1}\chi)(2^j y)u(x-y)\,\mathrm{d}y.$$

Lemma 2.5.3. *The operators* $\dot\Delta_j$ *and* $\dot S_j$ *are bounded* $L^p \to L^p$ *for all* $p \in [1,\infty]$*, with bounds independent of* j*.*

Proof. Exercise 2.6.12. □

Note that $\dot\Delta_j$ and $\dot S_j$ are *Fourier multipliers*, and as such they commute with other Fourier multipliers, like convolutions, derivatives, ...

The *formal* homogeneous Littlewood-Paley decomposition is

$$\mathrm{Id} = \sum_{j\in\mathbb{Z}} \dot\Delta_j,$$

but in what sense the series converges is, for now, unclear.

Definition 2.5.4 (Homogeneous Besov norms)**.** Let $s \in \mathbb{R}$ and $p, r \in [1,\infty]$. For any $u \in \mathcal{S}_0$ we define

$$\|u\|_{\dot B^s_{p,r}} := \Big(\sum_{j\in\mathbb{Z}} 2^{rjs}\|\dot\Delta_j u\|^r_{L^p}\Big)^{\frac{1}{r}}.$$

We call $\|\cdot\|_{\dot B^s_{p,r}}$ the *homogeneous Besov norm.*

Remark 2.5.5. We can think of the homogeneous Besov norms as follows. For each $j \in \mathbb{Z}$, take the L^p norm of $\dot\Delta_j u$, multiply it by 2^{js} and take the l^r norm of the resulting sequence.

Remark 2.5.6. One can check that, up to a constant, the definition of the Besov norm does not depend on the choice of the function ψ.

Remark 2.5.7. One also defines *homogeneous Besov spaces*, but there are some functional-theoretic subtleties which we would like to avoid here.

Proposition 2.5.8 (Duality for Besov norms)**.** *For any $s \in \mathbb{R}$ and $p, r \in [1, \infty]$ there exists $C \geq 0$ such that*

$$\langle \phi, u \rangle \leq C \|\phi\|_{\dot{B}^{-s}_{p',r'}} \|u\|_{\dot{B}^{s}_{p,r}}, \qquad \forall u, \phi \in \mathcal{S}_0$$

and

$$\|u\|_{\dot{B}^{s}_{p,r}} \leq C \sup_{\phi \in Q^{-s}_{p',r'}} \langle \phi, u \rangle, \qquad \forall u \in \mathcal{S}_0,$$

where $Q^{-s}_{p',r'}$ is the set of $\phi \in \mathcal{S}_0$ such that $\|\phi\|_{\dot{B}^{-s}_{p',r'}} \leq 1$.

Proof. □

Proposition 2.5.9. *For any $p \in [2, \infty)$ there exists C_p such that for all $u \in \mathcal{S}_0$*

$$\|u\|_{L^p} \leq C_p \|u\|_{\dot{B}^0_{p,2}}.$$

For any $p \in (1, 2]$ there exists C_p such that for all $u \in \mathcal{S}_0$

$$\|u\|_{\dot{B}^0_{p,2}} \leq C_p \|u\|_{L^p}.$$

Remark 2.5.10. This result is a part of the Littlewood-Paley theorem, a fundamental result in harmonic analysis, which is more difficult and hopefully we will not need it.

Proof of Proposition 2.5.9. □

Proposition 2.5.11 (Refined Sobolev inequality)**.** *Let $0 < s < \frac{d}{2}$ and $p = \frac{2d}{d-2s}$. Then*

$$\|f\|_{L^p} \leq C \|f\|^{\frac{p-2}{p}}_{\dot{B}^s_{2,\infty}} \|f\|^{\frac{2}{p}}_{\dot{H}^s}.$$

Proof. □

2.6 Exercises

Exercise 2.6.1. Prove the following special case of the Marcinkiewicz interpolation theorem. Let (X, μ) be a measure space and let T be a sublinear positive operator, that is an operator satisfying

$$f \geq 0 \ \Rightarrow \ Tf \geq 0, \qquad \text{for all measurable } f$$
$$T(af + bg) \leq aTf + bTg, \qquad \forall a, b \geq 0 \text{ and measurable positive } f, g.$$

Suppose moreover that T is bounded $L^1 \to L^1_w$ and $L^\infty \to L^\infty$, in other words there exist constants $C_1, C_\infty > 0$ such that

$$\sup_{\lambda > 0} \lambda \cdot \mu\{Tf > \lambda\} \leq C_1 \|f\|_{L^1},$$
$$\|Tf\|_{L^\infty} \leq C_\infty \|f\|_{L^\infty}.$$

Then T is bounded $L^p \to L^p$ for all $p \in (1, \infty)$.
Hint.

- Show that $T(0) = 0$ and $T(\lambda f) = \lambda T f$ for all $\lambda > 0$.
- It suffices to prove that there exists $C > 0$ such that for all f, g satisfying $\|f\|_{L^p} \leq 1$ and $\|g\|_{L^{p'}} \leq 1$ there is $\langle Tf, g\rangle \leq C$.
- Let $f = \sum_j c_j f_j$ and $g = \sum_k d_k g_k$ be atomic decompositions of f and g. Let $a_{jk} := \langle Tf_j, g_k\rangle$. Thus $\langle Tf, g\rangle \leq \sum_{(j,k)\in\mathbb{Z}^2} a_{jk} c_j d_k$. Using the Young inequality for the counting measure, prove that it is sufficient to show that $a_{jk} \leq A(j-k)$ for some summable function $A : \mathbb{Z} \to \mathbb{R}_+$.
- We will prove that there exist $\widetilde{C}, \epsilon > 0$ (depending on C_1 and C_∞) such that $A(n) = \widetilde{C} 2^{-\epsilon|n|}$ works. We treat separately $j \geq k$ and $k \geq j$.
- If $j \geq k$, use $\|g_k\|_{L^1} \leq 2^{-\frac{k}{p'}} 2^{k+1}$, $\|Tf_j\|_{L^\infty} \leq C_\infty 2^{-\frac{j}{p}}$ and conclude.
- In the case $j \leq k$, choose some $a \in (1, p)$, and then prove and use the following bounds:

$$\|g_k\|_{L^{a'}} \leq 2^{-\frac{k}{p'}} 2^{\frac{k+1}{a'}},$$

$$\|Tf_j\|_{L^a}^a \leq \frac{a}{a-1} C_1 C_\infty^{a-1} \|f_j\|_{L^1} \|f_j\|_{L^\infty}^{a-1} \lesssim 2^{-\frac{j}{p}+(j+1)-\frac{j}{p}(a-1)} \Rightarrow \|Tf_j\|_{L^a} \lesssim 2^{-\frac{j}{p}} 2^{\frac{j+1}{a}}.$$

Exercise 2.6.2. Prove (2.3.2).
Hint. Reduce to $\epsilon = 1$ and $d = 1$. Define $I(x) := \int_{\mathbb{R}} \mathrm{e}^{ix\xi - \frac{1}{2}|\xi|^2}\,\mathrm{d}\xi$. The value of $I(0)$ is well-known. There are at least two ways to conclude the proof:

- either use complex analysis to show that $e^{\frac{|x|^2}{2}} I(x)$ is independent of x,
- or check that $I'(x) = -xI(x)$ for all $x \in \mathbb{R}$.

Exercise 2.6.3. Prove the following generalisation of (2.3.2). Let $z \in \mathbb{C} \setminus \{0\}$ with $\Re z \geq 0$. Then

$$\mathcal{F}\big(\mathrm{e}^{-\frac{z}{2}|x|^2}\big) = \Big(\frac{2\pi}{z}\Big)^{\frac{d}{2}} \mathrm{e}^{-\frac{|\xi|^2}{2z}},$$

where $z^{-\frac{d}{2}} := |z|^{-\frac{d}{2}} \mathrm{e}^{-i\frac{d}{2}\theta}$ for $z = |z|\mathrm{e}^{i\theta}$ with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
Hint. For $\Re z > 0$, this follows from the unique continuation principle in complex analysis. In order to treat the case $\Re z$, use the fact that $\mathcal{F}$ is continuous $\mathcal{S}' \to \mathcal{S}'$.

Exercise 2.6.4. Prove Proposition 2.3.5.
Hint. Assuming the conclusion is false, construct a sequence $u_n \in \mathcal{S}(\mathbb{R}^d)$ such that $u_n \to 0$ in $\mathcal{S}(\mathbb{R}^d)$ and $\langle \phi, u_n\rangle \geq 1$ for all n.

Exercise 2.6.5. Prove Proposition 2.3.8.

Exercise 2.6.6. Prove Proposition 2.3.9.

Exercise 2.6.7. Prove Proposition 2.3.10.
Hint. Denote $\phi := \mathcal{F}(|x|^{-\alpha})$. Prove that ψ is homogeneous of degree $\alpha - d$ (which means $\langle \phi, u(\lambda\cdot)\rangle = \lambda^{-\alpha}\langle \phi, u\rangle$ for any $\lambda > 0$ and $u \in \mathcal{S}$) and rotationally symmetric (which means $\langle \phi, u(R\cdot)\rangle = \langle \phi, u\rangle$ for any rotation R and $u \in \mathcal{S}$). Set $\psi(\xi) := |\xi|^{d-\alpha}\phi(\xi)$ and deduce that ψ is homogeneous of degree 0 and rotationally symmetric. Show that $x \cdot \nabla\psi = 0$ and $(x_j\partial_k - x_k\partial_j)\psi = 0$ for $j \neq k$ in the distributional sense. Taking an appropriate linear combination deduce that $\langle \nabla\psi, u\rangle = 0$ for all $u \in \mathcal{S}_0$. Deduce that $\partial_j\psi$ is a polynomial for all j. One should be able to conclude from here, but to be honest at the moment I'm not sure how.

Exercise 2.6.8. Prove Proposition 2.3.12.
Hint. First prove, using Fubini's theorem, that for any $f, g \in \mathcal{S}$ we have $\int_{\mathbb{R}^d} f(\xi)\widehat{g}(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^d} \widehat{f}(x)g(x)\,\mathrm{d}x$.

Exercise 2.6.9. Prove Lemma 2.4.2.

Exercise 2.6.10. Show that $\mathcal{S} \subset \dot{H}^s$ if and only if $s > -\frac{d}{2}$. What about $\dot{B}^s_{p,r}$ instead of $\dot{H}^s$?

Exercise 2.6.11. Complete the proof of Proposition 2.4.4.

Exercise 2.6.12. Prove Lemma 2.5.3.

Exercise 2.6.13. Show that for any $u \in \mathcal{S}_0$ and any $s \in \mathbb{R}, p \in [1, \infty], r \in [1, \infty]$ the $\dot{B}^s_{p,r}$ norm of u is finite.

Chapter 3

Strichartz estimates

3.1 The linear wave propagator

We consider the linear wave equation without potential from $\mathbb{R}^d$ to $\mathbb{R}$:

$$\partial_t^2 u(t,x) = \Delta u(t,x), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d,\ u(t,x) \in \mathbb{R}.$$

Notice that there is no loss of generality in considering $u(t,x) \in \mathbb{R}$ instead of $u(t,x) \in \mathbb{R}^m$, because in the vector-valued case the components $u^{(j)}$ are decoupled.

We rewrite this equation in a standard way as a first-order in time system:

$$\partial_t \begin{pmatrix} u(t,x) \\ \dot{u}(t,x) \end{pmatrix} = \begin{pmatrix} \dot{u}(t,x) \\ \Delta u(t,x) \end{pmatrix}, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d,\ u(t,x), \dot{u}(t,x) \in \mathbb{R}. \tag{3.1.1}$$

We will write $\boldsymbol{u} := (u, \dot{u})$.

Definition 3.1.1. Let $\boldsymbol{u} = (u, \dot{u}) \in C([0,T], \mathcal{S}' \times \mathcal{S}')$. We say that $\boldsymbol{u}$ is a *weak solution* of (3.1.1) if for all $\boldsymbol{\phi} = (\phi, \dot{\phi}) \in C^\infty([0,T], \mathcal{S})$

$$\int_0^T \langle \dot{\phi}, u - \dot{u} \rangle + \langle \phi,$$

Proposition 3.1.2. *Let $s < \frac{d}{2}$. Denote $\mathcal{H}^s := \dot{H}^s \times \dot{H}^{s-1}$ and $\|\boldsymbol{u}_0\|_{\mathcal{H}^s} := \sqrt{\|u_0\|_{\dot{H}^s}^2 + \|\dot{u}_0\|_{\dot{H}^{s-1}}^2}$. For all $\boldsymbol{u}_0 = (u_0, \dot{u}_0) \in \mathcal{H}^s$ and $t_0 \in \mathbb{R}$ there exists a unique weak solution $\boldsymbol{u} \in C(\mathbb{R}, \mathcal{S}' \times \mathcal{S}')$ of* (3.1.1) *such that $\boldsymbol{u}(t_0) = \boldsymbol{u}_0$. This solution satisfies:*

$$\boldsymbol{u} \in C(\mathbb{R}, \mathcal{H}^s),$$
$$\|\boldsymbol{u}(t)\|_{\mathcal{H}^s} = \|\boldsymbol{u}_0\|_{\mathcal{H}^s}, \qquad \text{for all } t \in \mathbb{R}.$$

Let γ

We write $\boldsymbol{u}(t) = S(t,t_0)\boldsymbol{u}_0$. Thus $S(t,t_0)$ is an isometry of $\mathcal{H}^s$ for all t, t_0 and $s < \frac{d}{2}$.

We also consider the non-homogeneous equation

$$\partial_t^2 u(t,x) = \Delta u(t,x) + f(t,x), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d,\ u(t,x) \in \mathbb{R}. \tag{3.1.2}$$

Proposition 3.1.3 (Energy estimate)**.** *Let $s < \frac{d}{2}$. For all $\boldsymbol{u}_0 = (u_0, \dot{u}_0) \in \mathcal{H}^s$, $f \in L^1(I, \dot{H}^{s-1})$ and $t_0 \in I$ there exists a unique weak solution $\boldsymbol{u} \in C(\mathbb{R}, \mathcal{S}' \times \mathcal{S}')$ of* (3.1.2) *such that $\boldsymbol{u}(t_0) = \boldsymbol{u}_0$. This solution satisfies:*

$$\boldsymbol{u} \in C(I, \mathcal{H}^s),$$

$$\|\boldsymbol{u}(t)\|_{\mathcal{H}^s} \leq \|\boldsymbol{u}_0\|_{\mathcal{H}^s} + \Big|\int_{t_0}^{t} \|f(t')\|_{\dot{H}^{s-1}}\,\mathrm{d}t'\Big|, \qquad \textit{for all } t \in I.$$

Moreover, if $\boldsymbol{u}_0(x) = 0$ for $|x - x_0| \leq |t - t_0| + R$ and $f(t', x) = 0$ for $|x - x_0| \leq |t - t'| + R$, then $\boldsymbol{u}(t, x) = 0$ for $|x - x_0| \leq R$.

Remark 3.1.4. The last property is the *finite speed of propagation.*

Proof. We only treat the case of smooth data.

In order to prove the finite speed of propagation, we consider the vector field in $\mathbb{R}^{1+d}$

$$G(t, x) := \Big(\frac{1}{2}\big((\partial_t u)^2 + |\nabla u|^2\big), -\partial_t u \nabla u\Big).$$

We compute

$$\operatorname{div}_{\mathbb{R}^{1+d}} G(t, x) = \partial_t^2 u \partial_t u + \partial_t \nabla u \cdot \nabla u - \partial_t \nabla u \cdot \nabla u - \partial_t u \Delta u = f \partial_t u.$$

Without loss of generality take $t_0 = 0$, $x_0 = 0$ and $t \geq 0$. We apply the space-time divergence theorem to the cone bounded by the disks $D((0, x), R + |t|)$ and $D((t, x), R)$. We obtain the so-called *energy identity*:

$$\int_K f \partial_t u\,\mathrm{d}x = -\int_{|x| \leq R + |t|} \frac{1}{2}\big((\dot{u}_0)^2 + |\nabla u_0|^2\big)\,\mathrm{d}x + \int_{|x| \leq R} \frac{1}{2}\big((\partial_t u(t))^2 + |\nabla u(t)|^2\big)\,\mathrm{d}x + \frac{1}{2\sqrt{2}} \int_M |\nabla^{\perp} u|^2\,\mathrm{d}\sigma,$$

where K is the cone, M is the "side" of the cone and $\nabla^{\perp}$ is the tangential derivative. If the first two terms are identically zero, then the other two as well, which proves the claim. ☐

We can solve explicitly (3.1.2) by taking the Fourier transform in space variables. We obtain

$$\widehat{u}(t, \xi) = \widehat{u}_0(\xi) \cos(|\xi|(t - t_0)) + \widehat{\dot{u}_0} \frac{\sin(|\xi|(t - t_0))}{|\xi|} + \int_{t_0}^{t} \widehat{f}(s, \xi) \frac{\sin(|\xi|(t - s))}{|\xi|}\,\mathrm{d}s,$$

or equivalently

$$u(t) = \cos((t - t_0)|D|) u_0 + \frac{\sin((t - t_0)|D|)}{|D|} \dot{u}_0 + \int_{t_0}^{t} \frac{\sin((t - s)|D|)}{|D|} f(s)\,\mathrm{d}s.$$

We are led to study dispersive properties of the *half wave propagators* $\mathrm{e}^{\pm it|D|}$. Note that we can transform the wave equation to the half-wave equation formally by taking $u_+(t) := u(t) + \frac{1}{i|D|}\partial_t u(t)$. Denote

$$\langle x \rangle := \sqrt{1 + x^2}.$$

Proposition 3.1.5. *There exists $C \geq 0$ such that for all complex-valued $f \in \mathcal{S}$ such that $\operatorname{supp} \widehat{f} \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$ and all $t \in \mathbb{R}$*

$$\|\mathrm{e}^{\pm it|D|} f\|_{L^\infty} \leq C \langle t \rangle^{-\frac{1}{2}} \|f\|_{L^1}. \tag{3.1.3}$$

Proof. Without loss of generality take the sign "+". Let $\chi(\xi) = \chi(|\xi|) \in C^\infty$ be equal to 1 for $\frac{1}{2} \leq |\xi| \leq 2$ and to 0 for $|\xi| \leq \frac{1}{4}$ or $\xi \geq 4$. By our assumption, we have

$$\mathrm{e}^{it|D|} f = \mathrm{e}^{it|D|}\chi(|D|)f.$$

Taking the inverse Fourier transform, up to a normalising factor we get

$$\left(\mathrm{e}^{it|D|} f\right)(x) = (K_t * f)(x),$$

where

$$K_t(x) := \int_{\mathbb{R}^d} \mathrm{e}^{it|\xi| + i\xi\cdot x}\chi(\xi)\,\mathrm{d}\xi.$$

Thus it suffices to show that

$$\|K_t\|_{L^\infty} \lesssim \langle t\rangle^{-\frac{d-1}{2}}.$$

Changing to polar coordinates, we find

$$K_t(x) = \int_0^\infty \mathrm{e}^{itr}\chi(r)r^{d-1}\mathcal{F}^{-1}\sigma(rx)\,\mathrm{d}r = \int_0^\infty \mathrm{e}^{ir(t\pm|x|)}\chi(r)r^{d-1}\omega_\pm(rx)\,\mathrm{d}r,$$

where σ is the surface measure of $\mathbb{S}^{d-1}$ and we have used Corollary 2.4.8. If $\frac{1}{2}t \leq |x| \leq 2t$, the conclusion follows directly from Corollary 2.4.8. If not, we integrate by parts. $\square$

From Plancherel we have $\|\mathrm{e}^{\pm it|D|}\gamma\|_{L^2} = \|\gamma\|_{L^2}$, so (3.1.3) and Riesz-Thorin theorem yield

$$\|\mathrm{e}^{\pm it|D|}\gamma\|_{L^q} \leq C\langle t\rangle^{-\frac{1}{2}(\frac{1}{p'}-\frac{1}{p})t}\|\gamma\|_{L^{p'}}, \qquad \forall f \in \mathcal{S},\ p \in [2,\infty].$$

3.2 The TT^* method

In this section, we prove general Strichartz estimates. For f a measurable function on $\mathbb{R}\times\mathbb{R}^d$ and $p, q \in [1,\infty]$ we define

$$\|f\|_{L^pL^q} := \left(\int_{\mathbb{R}} \|f(t,\cdot)\|_{L^q}^p\,\mathrm{d}t\right)^{\frac{1}{p}}.$$

Measurability.

Lemma 3.2.1. *Let $(p_j, q_j) \in [1,\infty]^2$ and $\theta_j \geq 0$ with $\sum_{j=1}^m \theta_j = 1$. Suppose that*

$$\frac{1}{p} = \sum_{j=1}^m \frac{\theta_j}{p_j}, \qquad \frac{1}{q} = \sum_{j=1}^m \frac{\theta_j}{q_j}.$$

Then

$$\|f\|_{L^pL^q} \leq \prod_{j=1}^m \|f\|_{L^{p_j}L^{q_j}}^{\theta_j}, \qquad \forall f \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^d).$$

Proof. Exercise. $\square$

Definition 3.2.2. Let $\sigma > 0$. We say that a pair (p, q) is *σ-admissible* if

$$\frac{1}{p} + \frac{\sigma}{q} = \frac{\sigma}{2}, \qquad (p, q, \sigma) \neq (2, \infty, 1).$$

If σ is known from the context, we can call such a pair *admissible*.

Remark 3.2.3. It is easy to see that in the case $\sigma = 0$ we do not obtain anything interesting. We would be forced to admit $(\infty, 2)$ is the only 0-admissible pair.

Theorem 3.2.4. *Let $U(t)$ be a bounded family of continuous operators such that*

$$\|U(t)U^*(t')f\|_{L^\infty} \leq C|t - t'|^{-\sigma}\|f\|_{L^1}, \qquad \forall t, t' \in \mathbb{R},\ f \in \mathcal{S}. \tag{3.2.1}$$

Let $\chi : \mathbb{R}^2 \to \mathbb{C}$ be a measurable function such that $|\chi(t, t')| \leq 1$ for all t, t'. Then for all σ-admissible pairs (p, q)

$$\left\| \int_{\mathbb{R}} \chi(t, t')U(t)U^*(t')f(t')\,\mathrm{d}t' \right\|_{L^{p_1}L^{q_1}} \leq C\|f\|_{L^{p_2'}L^{q_2'}}, \tag{3.2.2}$$

with C independent of χ.

Proof of Theorem 3.2.4 in the non-endpoint case. The proof is considerably easier in the non-endpoint case $p > 2$, so we present it first.
Step 1. For $f, g \in C^\infty(\mathbb{R}, \mathcal{S})$ we define

$$T_\chi(f, g) := \int_{\mathbb{R}^2} \chi(t, t')\langle U(t)U^*(t')f(t'), g(t)\rangle\,\mathrm{d}t\,\mathrm{d}t',$$

where $\langle\cdot, \cdot\rangle$ is the L^2 inner product. By duality, (3.2.2) is equivalent to

$$|T_\chi(f, g)| \leq C\|f\|_{L^{p_2'}L^{q_2'}}\|g\|_{L^{p_1'}L^{q_1'}}. \tag{3.2.3}$$

Step 2. We show (3.2.3) with $(p_2, q_2) = (p_1, q_1)$. Interpolating between (3.2.1) and the $L^2 \to L^2$ bound we have

$$\|U(t)U^*(t')f(t')\|_{L^q} \leq |t - t'|^{-\sigma\left(1-\frac{2}{q}\right)}\|f(t')\|_{L^{q'}},$$

thus

$$\langle U(t)U^*(t')f(t'), g(t)\rangle \leq C|t - t'|^{-\sigma\left(1-\frac{2}{q}\right)}\|f(t')\|_{L^{q'}}\|g(t)\|_{L^{q'}} = C|t - t'|^{-\frac{2}{p}}\|f(t')\|_{L^{q'}}\|g(t)\|_{L^{q'}},$$

and we conclude using Hardy-Littlewood-Sobolev inequality, using the fact that $2 < p < \infty$.
Step 3. We prove that

$$\left\| \int_{\mathbb{R}} U^*(t)f(t)\,\mathrm{d}t \right\|_{L^2} \leq C\|f\|_{L^{p'}L^{q'}}. \tag{3.2.4}$$

Denote $T = T_\chi$ with $\chi(t, t') = 1$ for all t, t'. Directly from the definition of T_χ we obtain

$$T(f, f) = \left\| \int_{\mathbb{R}} U^*(t)f(t)\,\mathrm{d}t \right\|_{L^2}^2,$$

so (3.2.4) follows from Step 1.
Step 4. We prove (3.2.3) for any σ-admissible pairs (p_1, q_1) and (p_2, q_2). By symmetry, without loss

of generality we can assume $q_1 \leq q_2$. Fixing t and using (3.2.4) with t' instead of t and $\chi(t,t')f(t')$ instead of $f(t)$ we get

$$\Big\| \int_{\mathbb{R}} \chi(t,t')U(t)U^*(t')f(t')\,\mathrm{d}t' \Big\|_{L^\infty L^2} \leq C\|f\|_{L^{p_2'}L^{q_2'}}.$$

Lemma 3.2.1 and (3.2.2) for $(p_1,q_1) = (p_2,q_2)$ thus imply (3.2.2) in the general case. □

The endpoint case $p = 2$ is much more difficult. It was first settled by Keel and Tao [5]. The Hardy-Littlewood-Sobolev inequality is not directly applicable. Instead, we will revisit its proof in our particular setting. First, we write

$$T_\chi(f,g) = \sum_{j\in\mathbb{Z}} T_j(f,g) := \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^2} \chi_j(t,t')\langle U(t)U^*(t')f(t'), g(t)\rangle\,\mathrm{d}t\,\mathrm{d}t',$$

where $\chi_j(t,t') := I_{2^j\leq|t-t'|<2^{j+1}}\chi(t,t')$. Our main goal is to prove (3.2.3) with $p_1 = p_2 = 2$ and $q_1 = q_2 = q = \frac{2\sigma}{\sigma-1} < \infty$.

Lemma 3.2.5. *There exists an open neighbourhood V of (q,q) in $\mathbb{R}^2$ such that for all $(a,b)\in V$ and $j\in\mathbb{Z}$*

$$|T_j(f,g)| \leq C2^{-j\beta(a,b)}\|f\|_{L^2L^{a'}}\|g\|_{L^2L^{b'}}, \quad \beta(a,b) := \sigma - 1 - \frac{\sigma}{a} - \frac{\sigma}{b}. \tag{3.2.5}$$

Proof of Theorem 3.2.4 in the endpoint case, assuming Lemma 3.2.5. The proof is based on the atomic decomposition lemma. □

Proof of Lemma 3.2.5. Considering $\widetilde{U}(t) := U(2^jt)$, $\widetilde{f}(t,x) := f(2^jt, 2^{\sigma j}x)$ and $\widetilde{g}(t,x) := g(2^jt, 2^{\sigma j}x)$ we reduce to $j = 0$.
Step 1. We prove (3.2.5) for $a = b = \infty$. This easily follows from (3.2.1).
Step 2. Using the non-endpoint case, we show that (3.2.5) holds for $b = 2$ and $2 \leq a < q$, as well as for $a = 2$ and $2 \leq b < q$. **Step 3.** We use interpolation. What exactly interpolation theorem are we using?

□

3.3 Strichartz estimates for the wave equation

Definition 3.3.1. We say that a pair (p,q) is *wave-admissible* if there exists $2 \leq \widetilde{q} \leq q$ such that

$$\frac{2}{p} + \frac{d-1}{\widetilde{q}} = \frac{d-1}{2}, \qquad (p,\widetilde{q},d) \neq (2,\infty,3).$$

Theorem 3.3.2. *Suppose that (p,q) and (a,b) are wave-admissible and*

$$\frac{1}{p} + \frac{d}{q} = \frac{1}{a'} + \frac{d}{b'} - 2 = \frac{d}{2} - \sigma.$$

Let u be the solution of (3.1.2). *Then*

$$\|u\|_{L^pL^q} \leq C\big(\|\boldsymbol{u}_0\|_{\mathcal{H}^\sigma} + \|f\|_{L^{a'}L^{b'}}\big).$$

We first prove that the theorem is true if all the functions involved have spatial Fourier transforms contained in $\{\frac{1}{2} \leq |\xi| \leq 2\}$. This is done using Theorem 3.2.4.

By scaling invariance, this implies that the conclusion holds if all the functions involved have spatial Fourier transforms contained in $\{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ for some $j \in \mathbb{Z}$.

The third step is to "glue the Littlewood-Paley pieces", which we are now going to explain.

Note that $\dot{\Delta}_j$ commutes with $\mathrm{e}^{it|D|}$. Thus

$$\|\dot{\Delta}_j \mathrm{e}^{it|D|} f\|_{L^p L^q} \leq C \|\dot{\Delta}_j f\|_{\dot{H}^\sigma}. \tag{3.3.1}$$

For fixed t we can write:

$$\|\mathrm{e}^{it|D|} f\|^2_{\dot{B}^0_{q,2}} = \sum_{j \in \mathbb{Z}} \|\dot{\Delta}_j \mathrm{e}^{it|D|} f\|^2_{L^q},$$

so the Minkowski inequality and (3.3.1) yield

$$\|\mathrm{e}^{it|D|} f\|_{L^p \dot{B}^0_{q,2}} \leq C \|f\|_{\dot{H}^\sigma}.$$

Finally, we use $\dot{B}^0_{q,2} \subset L^q$.

Chapter 4

Cauchy theory for wave equations

4.1 ODE in Banach spaces

Theorem 4.1.1. *Let E be a Banach space, Ω an open subset of E, I an open interval and $(t_0, x_0) \in I \times \Omega$. Let $F \in C(I, \mathrm{Lip}(\Omega; E))$. There exists $J \subset I$ such that the equation*

$$x(t) = x_0 + \int_{t_0}^{t} F(t', x(t'))\,\mathrm{d}t' \tag{4.1.1}$$

has a unique continuous solution on J. Moreover, this solution is continuous with respect to the initial data.

Proof. For $x(t)$ a continuous function on $[t_0 - \epsilon, t_0 + \epsilon]$ with values in Ω we define

$$\Phi(x)(t) := x_0 + \int_{t_0}^{t} F(t', x(t'))\,\mathrm{d}t'.$$

We see that $\Phi(x)(t)$ is also a continuous function on $[t_0 - \epsilon, t_0 + \epsilon]$ with values in Ω if ϵ is small enough, and it is a contraction. The conclusion follows from the Uniform Contraction Principle (with t_0 and x_0 as parameters). □

For a solution of (4.1.1) we can define its *maximal interval of existence* (T_-, T_+), with $-\infty \le T_- < t_0 < T_+ \le \infty$.

Proposition 4.1.2. *If $T_+ < \infty$, then $\lim_{t \to T_+} \|x(t)\| = \infty$.*

Proof. □

4.2 Cauchy problem by the energy method

Consider the equation (1.2.1), which we write as follows:

$$\partial_t^2 u = \Delta u + f(x, u).$$

We assume that $d \geq 3$ for some integer $s > \frac{d}{2}$ the derivatives $\partial_{x,u}^j f(x,u)$ are bounded for $0 \leq j \leq s$ (locally with respect to u and globally with respect to x) and $f(x,0) = 0$. We consider the initial value problem

$$\begin{aligned} \partial_t^2 u &= \Delta u + f(x,u), \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0. \end{aligned} \tag{4.2.1}$$

We would like to prove existence and uniqueness for smooth enough initial data, but the general ODE theory does not directly apply, because the Laplacian is not bounded on any reasonable space. There are several remedies. Probably the simplest one is to solve the non-homogeneous wave equation instead of just integrating in time.

Proposition 4.2.1. *For any $\boldsymbol{u}_0 \in X^s := (\dot{H}^s \cap \dot{H}^1) \times H^{s-1}$ the initial value problem (4.2.1) has a unique solution $\boldsymbol{u}(t) \in C^1((T_-, T_+), X^s)$. Moreover, if $T_+ < \infty$, then $\lim_{t \to T_+} \|\boldsymbol{u}(t)\|_{X^s} = \infty$. Finally, finite speed of propagation holds.*

Proof. □

Remark 4.2.2. If f is analytic with respect to u, then the solution map is analytic with respect to t_0 and $\boldsymbol{u}_0$.

Remark 4.2.3. It is also possible to use explicit representation formulas for the wave equation to prove existence of solutions in $C^k \times C^{k-1}$ with appropriate k for sufficiently regular initial data.

4.3 Low regularity

We will need to work with solutions having less regularity. We take inspiration from the last proof to define what a solution means.

Definition 4.3.1. Let $0 \leq s < \frac{d}{2}$. We say that a function $\boldsymbol{u} \in L^\infty(I, \mathcal{H}^s)$ solves the Cauchy problem (4.2.1) on an open interval I if for any closed $J \subset I$:

1. $f(x, u(t,x)) \in L^1(J; \dot{H}^{s-1})$,
2. $u(t,x)$ solves the non-homogeneous wave equation in the sense of formula (3.1.2).

As an example, consider $f(x,u) = \pm|u|^{p-1}u$ with $3 \leq p < 5$ in dimension $d = 3$.

Proposition 4.3.2. *For all $\boldsymbol{u}_0 \in \mathcal{H}^1(\mathbb{R}^3)$ the equation*

$$\partial_t^2 u = \Delta u \pm |u|^{p-1} u \tag{4.3.1}$$

has a unique solution $\boldsymbol{u} \in C((T_-, T_+), \mathcal{H}^1)$. It is continuous with respect to the initial data $\boldsymbol{u}_0$. There is finite speed of propagation. Moreover, if $T_+ < \infty$, then $\lim_{t \to T_+} \|\boldsymbol{u}(t)\|_{\mathcal{H}^1} = \infty$.

Remark 4.3.3. We usually express the first part of the proposition by saying that the equation is *locally well-posed* in $\mathcal{H}^1$.

Proof. Without loss of generality assume $t_0 = 0$ and denote $\boldsymbol{u}_L$ the solution of the linear homogeneous wave equation with initial data $\boldsymbol{u}_L(0) = \boldsymbol{u}_0$. We define the functional Φ as follows. If $\boldsymbol{u}(t)$ is any function in the ball $B(\boldsymbol{u}_L, \rho)$ Define the norm, then

$$\Phi(\boldsymbol{u})(t) := \boldsymbol{u}_L(t) + \int_0^t \frac{\sin((t-s)|D|)}{|D|}\big(|u(s)|^{p-1}u(s)\big)\,\mathrm{d}s.$$

We should prove that this is a contraction. This relies on the following inequality:

$$\|u_L\|_{L^pL^{2p}} \leq \|u_L\|_{L^3L^6}^{\alpha}\|u_L\|_{L^5L^{10}}^{1-\alpha} \lesssim T^\beta\|\boldsymbol{u}_0\|_{\mathcal{H}^1}, \qquad \text{for some } \beta > 0.$$

□

The main reason to care about low regularity is to use conservation laws. Here is an example.

Lemma 4.3.4. *For $\boldsymbol{u}_0$ define*

$$E(\boldsymbol{u}_0) := \int_{\mathbb{R}^3} \frac{1}{2}\dot{u}_0^2 + \frac{1}{2}|\nabla u_0|^2 \mp \frac{1}{p+1}|u_0|^{p+1}.$$

Let $\boldsymbol{u}(t)$ be the solution constructed in the last Lemma. Then for all $t \in (T_-, T_+)$

$$E(\boldsymbol{u}(t)) = E(\boldsymbol{u}_0).$$

Proof. Approximate with smooth solutions and use continuity with respect to the initial data. □

Theorem 4.3.5. *Let $\boldsymbol{u}_0 \in \mathcal{S} \times \mathcal{S}$ and let $\boldsymbol{u}(t)$ be the solution of the Cauchy problem*

$$\partial_t^2 u = \Delta u - |u|^{p-1}u,$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0.$$

Then $T_- = -\infty$ and $T_+ = \infty$.

First proof. Our first proof only needs $\boldsymbol{u}_0 \in \mathcal{H}^1$. Since $\|\boldsymbol{u}(t)\|_{\mathcal{H}^1}^2 \leq 2E(\boldsymbol{u}(t))$, the conclusion follows from the conservation of energy. □

Second proof. The second proof is the original proof due to Jörgens [**?**], long before Strichartz estimates where invented.

□

4.4 Critical regularity

One often uses the following intuitive reasoning. Let $u(t,x)$ be a solution of (4.3.1) and $\lambda > 0$. Consider

$$u_{1/\lambda}(t,x) := \lambda^{\frac{2}{p-1}}u(\lambda t, \lambda x).$$

We easily see that u_λ also solves (4.3.1). If $\lambda \ll 1$, this means we are zooming in. An integration by parts yields

$$\|(\boldsymbol{u}_0)_{1/\lambda}\|_{\mathcal{H}^1} = \lambda^{\frac{5-p}{2(p-1)}}\|\boldsymbol{u}_0\|_{\mathcal{H}^1}, \qquad E((\boldsymbol{u}_0)_{1/\lambda}) = \lambda^{\frac{5-p}{p-1}}E(\boldsymbol{u}_0),$$

where it is understood that

$$(\boldsymbol{u}_0)_{1/\lambda} = (\lambda^{\frac{2}{p-1}} u_0(\lambda x), \lambda^{1+\frac{2}{p-1}} \dot{u}_0(\lambda x)).$$

Blow-up is supposed to be a small-scale phenomenon. But on the small scale we are dealing with small-energy solutions, so nothing "bad" should happen.

With an equation like (4.3.1) we can associate a *critical norm*. Its main purpose is to have a homogeneous (with respect to rescaling) functional which allows to meaningfully speak of "small data" for a given problem. Let $s_c := \frac{3}{2} - \frac{2}{p-1}$. We call s_c the *critical exponent*, $\|\cdot\|_{\mathcal{H}^s}$ the *critical norm* and $\mathcal{H}^s$ the *critical space*. If the space $\mathcal{H}^{s_c}$ is directly related to some important conserved quantity, the equation has a special name, for instance when $p = 5$ we have $s_c = 1$, which is related to the energy, and we say (4.3.1) for $p = 5$ is *energy-critical*.

For $\boldsymbol{u}_0 \in \mathcal{H}^s$ set

$$(\boldsymbol{u}_0)_\lambda(x) := (\lambda u_0(\lambda x), \lambda^{1+} \dot{u}_0(\lambda_x)).$$

It is easy to check that

$$\|(\boldsymbol{u}_0)_\lambda\|_{\mathcal{H}^s} = \|\boldsymbol{u}_0\|_{\mathcal{H}^s},$$

so that "small" or "large" does not depend on rescaling.

A phenomenon often related to small data is the so-called *scattering*.

Definition 4.4.1. Let $\boldsymbol{u}(t)$ be a solution of (4.2.1). We say that $\boldsymbol{u}(t)$ *scatters for positive times* in norm $\mathcal{H}^s$ if $T_+ = \infty$ and there exists a solution of the homogeneous linear wave equation $\boldsymbol{u}^-$ such that

$$\lim_{t\to\infty} \|\boldsymbol{u}(t) - \boldsymbol{u}^-(t)\|_{\mathcal{H}^s} = 0.$$

Theorem 4.4.2. *Assume* $3p^2 - 11p + 4 \geq 0$. *Equation* (4.2.1) *is locally well-posed in* $\mathcal{H}^{s_c}$. *Moreover, there exists* $\eta > 0$ *such that if* $\|\boldsymbol{u}_0\|_{\mathcal{H}^{s_c}} \leq \eta$, *then the corresponding solution is globally defined and scatters in both time directions.*

Proof. We use Strichartz estimates for the wave-admissible pairs $(p, \frac{3p(p-1)}{p+1})$ and $(\infty, \frac{3(p-1)}{2p-4})$. By a fixed point argument, we obtain a solution such that $|u|^{p-1}u \in L^1(\mathbb{R}, \dot{H}^{s_c-1})$. It follows that the solution scatters. $\square$

Remark 4.4.3. I think this is true also for smaller p, but I don't know how small.

4.5 Equivariant wave maps

We will study wave maps $\psi : \mathbb{R}^{1+2} \to \mathbb{S}^2 \subset \mathbb{R}^3$. Recall that they are critical points for the Lagrangian

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^2} \Big(\frac{1}{2}|\partial_t\psi|^2 - \frac{1}{2}|\nabla_x\psi|^2\Big)\,\mathrm{d}x.$$

The equation can be written explicitly:

$$\partial_t^2\psi - \Delta\psi = (|\partial_t\psi|^2 - |\nabla\psi|^2)\psi. \tag{4.5.1}$$

This equation is difficult to study. We will consider a particular class of solutions. Take $k \in \{1, 2, \ldots\}$ and consider initial data of the form

$$\psi_0(r\cos\theta, r\sin\theta) = \big(\sin(u_0(r))\cos(k\theta), \sin(u_0(r))\sin(k\theta), \cos(u_0(r))\big).$$

The evolution preserves this particular form of initial data and we obtain a simple equation for the scalar-valued function $u(t, r)$:

$$\partial_t^2 u = \partial_r^2 u + \frac{1}{r}\partial_r u - \frac{k^2 \sin(2u)}{2r^2}. \tag{4.5.2}$$

Until the end of the course, we will be concerned with equation (4.5.2), and perhaps with its generalization

$$\partial_t^2 u = \partial_r^2 u + \frac{1}{r}\partial_r u - \frac{f(u)}{r^2},$$

where $f = gg'$ for some smooth odd function $g : \mathbb{R} \to \mathbb{R}$.

There is the conserved energy given by

$$E(\boldsymbol{u}_0) := 2\pi \int_0^\infty \Big(\frac{1}{2}(\dot{u}_0)^2 + \frac{1}{2}(\partial_r u_0)^2 + \frac{g(u_0)^2}{2r^2}\Big) r\,\mathrm{d}r,$$

in the case of (4.5.2) given by

$$E(\boldsymbol{u}_0) := 2\pi \int_0^\infty \Big(\frac{1}{2}(\dot{u}_0)^2 + \frac{1}{2}(\partial_r u_0)^2 + \frac{k^2\sin(u_0)^2}{2r^2}\Big) r\,\mathrm{d}r.$$

We see that our problem is energy-critical.

For $m, n \in \mathbb{Z}$ we define

$$\mathcal{H}_{m,n} := \{\boldsymbol{u}_0 : E(\boldsymbol{u}_0) < \infty,\ \lim_{r\to 0} u(r) = m\pi,\ \lim_{r\to\infty} u(r) = n\pi\}.$$

Exercise 4.5.1. Prove that if $E(\boldsymbol{u}_0) < \infty$, then there exist $m, n \in \mathbb{Z}$ such that $\lim_{r\to 0} u_0(r) = m\pi$ and $\lim_{r\to\infty} u_0(r) = n\pi$.

The sets $\mathcal{H}_{m,n}$ are affine spaces and play the role of the critical space. We will mainly work in the space $\mathcal{H}_0 := \mathcal{H}_{0,0}$, which is a linear space. We define the critical norm:

$$\|\boldsymbol{u}_0\|_{\mathcal{H}_0}^2 := \int_0^\infty \Big(\dot{u}_0^2 + (\partial_r u_0)^2 + \frac{1}{r^2}u_0^2\Big) r\,\mathrm{d}r.$$

We also denote the part corresponding to the potential energy

$$\|u_0\|_H := \int_0^\infty \Big((\partial_r u_0)^2 + \frac{1}{r^2}u_0^2\Big) r\,\mathrm{d}r.$$

4.5.1 Review of the Cauchy theory

For the time being, this part is a copy-paste of my paper with Andy Lawrie. I need to work on it, so please handle with care.

For initial data (φ_0, φ_1) in the class $\mathcal{H}_0$ the formulation of the Cauchy problem (**??**) can be modified to take into account the strong repulsive potential term in the nonlinearity:

$$\frac{k^2 \sin(2\phi)}{2r^2} = \frac{k^2}{r^2}\phi + \frac{k^2}{2r^2}(\sin(2\phi) - 2\phi) = \frac{k^2}{r^2}\phi + \frac{O(\phi^3)}{r^2}$$

The presence of the potential $\frac{k^2}{r^2}$ indicates that the linear wave equation,

$$(\partial_t^2 - \Delta_{\mathbb{R}^2} + \frac{k^2}{r^2})\psi = 0, \tag{4.5.3}$$

of (**??**) has more dispersion than the $2d$ wave equation. In fact, it has the same dispersion as the free wave equation in dimension $d = 2k+2$ as can be seen from the following change of variables: given a radial function $\phi \in H$, define $v(r)$ by $\phi(r) = r^k v(r)$. Then

$$\frac{1}{r^k}(-\Delta_{\mathbb{R}^2} + \frac{k^2}{r^2})\phi = -\Delta_{\mathbb{R}^{2k+2}} v, \quad \|\phi\|_H = \|v\|_{\dot{H}^1(\mathbb{R}^{2k+2})}.$$

Thus one way of studying solutions $\vec{\psi}(t) \in \mathcal{H}_0$ of Cauchy problem (**??**) is to define $\vec{v}(t) = (r^{-k}\psi(t), r^{-k}\psi_t(t)) \in \dot{H}^1 \times L^2(\mathbb{R}^{2k+2})$ and analyse the equivalent Cauchy problem for the radial nonlinear wave equation in $\mathbb{R}^{1+(2k+2)}_{t,x}$ satisfied by $\vec{v}(t)$. Unfortunately, this route leads to unpleasant technicalities when $k > 2$ (spatial dimension $= 2k+2 > 6$) due to the high dimension and the particular structure of the nonlinearity.

There is a simpler approach that allows us to treat the scattering theory for the Cauchy problem (**??**) for all equivariance classes $k \geq 1$ in a unified fashion. The idea is to make use of some, but not all, of the extra dispersion in $-\Delta_{\mathbb{R}^2} + k^2/r^2$. Indeed, given a solution $\vec{\psi}(t)$ to (**??**) we define u by $ru = \psi$ and obtain the following Cauchy problem for u.

$$\begin{aligned} &\partial_t^2 u - \partial_r^2 u - \frac{3}{r}\partial_r u + \frac{k^2-1}{r^2}u = k^2\frac{2ru - \sin(2ru)}{2r^3} =: Z(ru)u^3 \\ &\vec{u}(0) = (u_0, u_1). \end{aligned} \tag{4.5.4}$$

where the function Z defined above is a clearly smooth, bounded, even function. The linear part of (4.5.4) is the radial wave equation in $\mathbb{R}^{1+4}$ with a *repulsive* inverse square potential, namely

$$v_{tt} - v_{rr} - \frac{3}{r}v_r + \frac{k^2-1}{r^2}v = 0. \tag{4.5.5}$$

For each $k \geq 1$, define the norm H_k for radially symmetric functions v on $\mathbb{R}^4$ by

$$\|v\|^2_{H_k(\mathbb{R}^4)} := \int_0^\infty \left[(\partial_r v)^2 + \frac{(k^2-1)}{r^2}v^2\right] r^3 \, \mathrm{d}r$$

Solutions to (4.5.5) conserve the H_k norms. By Hardy's inequality we have

$$\|v\|_{H_k(\mathbb{R}^4)} \simeq \|v\|_{\dot{H}^1(\mathbb{R}^4)}$$

The mapping,

$$H_k \times L^2(\mathbb{R}^4) \ni (u_0, u_1) \mapsto (\psi_0, \psi_1) := (ru_0, ru_1) \in H \times L^2(\mathbb{R}^2)$$

satisfies

$$\|(u_0, u_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^4)} \simeq \|(u_0, u_1)\|_{H_k \times L^2(\mathbb{R}^4)} = \|(\psi_0, \psi_1)\|_{H \times L^2(\mathbb{R}^2)}$$

Thus we can conclude that the Cauchy problem for (4.5.4) with initial data in $\dot{H}^1 \times L^2(\mathbb{R}^4)$ is equivalent to the Cauchy problem for (**??**) for initial data $(\psi_0, \psi_1) \in \mathcal{H}_0$, allowing us to give a scattering criterion for solutions $\vec{\psi}(t) \in \mathcal{H}_0$ to (**??**).

Lemma 4.5.2. *Let $\vec{\psi}(0) = (\psi_0, \psi_1) \in \mathcal{H}_0$. Then there exists a unique solution $\vec{\psi}(t) \in \mathcal{H}_0$ to* (**??**) *defined on a maximal interval of existence $I_{\max}(\vec{\psi}) := (-T_-(\vec{\psi}), T_+(\vec{\psi}))$ with the following properties: Define*

$$\vec{u}(t, r) = (r^{-1}\psi(t, r), r^{-1}\psi_t(t, r)) \in \dot{H}^1 \times L^2(\mathbb{R}^4)$$

Then for any compact time interval $J \Subset I_{\max}$ we have

$$\|u\|_{L^3_t(J; L^6_x(\mathbb{R}^4))} \leq C(J) < \infty$$

In addition, if

$$\|u\|_{L^3_t([0, T_+(\vec{\psi})); L^6_x(\mathbb{R}^4))} < \infty$$

then $T_+ = \infty$ and $\vec{\psi}(t)$ scatters $t \to \infty$, i.e., there exists a solution $\vec{\phi}_L(t) \in \mathcal{H}_0$ to (4.5.3) *so that*

$$\|\vec{\psi}(t) - \vec{\phi}_L(t)\|_{\mathcal{H}_0} \to 0 \qquad \text{as } t \to \infty.$$

Conversely, any solution $\vec{\psi}(t)$ that scatters as $t \to \infty$ satisfies

$$\|\psi/r\|_{L^3_t L^6_x([0, \infty) \times \mathbb{R}^4)} < \infty.$$

The proof of Lemma 4.5.2 is standard consequence of Strichartz estimates for (4.5.5) and the equivalence of the Cauchy problems (**??**) and (4.5.4). In this case, we need Strichartz estimates for the radial wave equation in $\mathbb{R}^{1+4}$ with a repulsive inverse square potential. For these we can cite the more general results of Planchon, Stalker, and Tahvildar-Zadeh [**?**]; see also [**?**, **?**] which cover the non-radial case.

Lemma 4.5.3 (Strichartz estimates)**.** [**?**, Corollary 3.9] *Fix $k \geq 1$ and let $\vec{v}(t)$ be a radial solution to the linear equation*

$$v_{tt} - v_{rr} - \frac{3}{r} v_r + \frac{k^2 - 1}{r^2} v = F(t, r), \quad \vec{v}(0) = (v_0, v_1) \in \dot{H}^1 \times L^2(\mathbb{R}^4)$$

Then, for any time interval $I \subset \mathbb{R}$ we have

$$\|v\|_{L^3_t L^6_x(I \times \mathbb{R}^4)} + \sup_{t \in I} \|\vec{v}(t)\|_{\dot{H}^1 \times L^2(\mathbb{R}^4)} \lesssim \|\vec{v}(0)\|_{\dot{H}^1 \times L^2(\mathbb{R}^4)} + \|F\|_{L^1_t, L^2_x(I \times \mathbb{R}^4)}$$

where the implicit constant above is independent of I.

We'll also explicitly require the following nonlinear perturbation lemma from [**?**]; see also [**?**, Lemma 2.18].

Lemma 4.5.4 (Perturbation Lemma). [?, Theorem 2.20] [?, Lemma 2.18] *There are continuous functions $\varepsilon_0, C_0 : (0,\infty) \to (0,\infty)$ such that the following holds: Let $I \subset \mathbb{R}$ be an open interval, (possibly unbounded), $\psi, \varphi \in C^0(I;H) \cap C^1(I;L^2)$ radial functions satisfying for some $A > 0$*

$$\|\vec{\psi}\|_{L^\infty(I;H\times L^2(\mathbb{R}^2))} + \|\vec{\varphi}\|_{L^\infty(I;H\times L^2(\mathbb{R}^2))} + \|\varphi/r\|_{L^3_t(I;L^6_x(\mathbb{R}^4))} \le A$$
$$\|eq(\psi/r)\|_{L^1_t(I;L^2_x(\mathbb{R}^4))} + \|eq(\varphi/r)\|_{L^1_t(I;L^2_x(\mathbb{R}^4))} + \|w_0\|_{L^3_t(I;L^6_x)} \le \varepsilon \le \varepsilon_0(A)$$

where $eq(\psi/r) := (\Box_{\mathbb{R}^4} + \frac{k^2-1}{r^2})(\psi/r) + (\psi/r)^3 Z(\psi)$ in the sense of distributions, and $\vec{w}_0(t) := S(t-t_0)(\vec{\psi} - \vec{\varphi})(t_0)$ with $t_0 \in I$ arbitrary, but fixed and S denoting the linear wave evolution operator in $\mathbb{R}^{1+4}$ (i.e., the propagator for (4.5.3)). Then,

$$\|\vec{\psi} - \vec{\varphi} - \vec{w}_0\|_{L^\infty_t(I;H\times L^2(\mathbb{R}^2))} + \|\frac{1}{r}(\psi - \varphi)\|_{L^3_t(I;L^6_x(\mathbb{R}^4))} \le C_0(A)\varepsilon$$

In particular, $\|\psi/r\|_{L^3_t(I;L^6_x(\mathbb{R}^4))} < \infty$.

Chapter 5

Profile decomposition

5.1 Abstract theory

We consider the following situation. Let H be a separable Hilbert space and G a topological group acting on H. We assume that G is locally compact and write $g_n \to \infty$ if for any compact $K \subset G$ we have $g_n \notin K$ for n large.

Definition 5.1.1. We say that u_n converges to 0 *weakly with concentration* if

$$g_n u_n \rightharpoonup 0, \qquad \forall g_n \in G.$$

We notice that the topology of weak with concentration convergence on a ball in H is metrisable as follows. Let ϕ_k be a dense sequence in the unit ball of H. We define

$$\|u\|_G := \sup_{g \in G} \left(\sum_{k=1}^{K} \frac{\langle \phi_k, gu \rangle^2}{2^k} \right)^{\frac{1}{2}}.$$

Definition 5.1.2. We say that two sequences g_n and $\widetilde{g}_n$ are *orthogonal* if $\lim_{n\to\infty} g_n^{-1}\widetilde{g}_n = \infty$.

Concerning the action of G on H, $L_g : H \to H$ (we could say, the *representation* of G), we assume

- $\|L_g\|$ bounded in the operator topology,
- $g_n \to \infty$ implies $L_{g_n} \to 0$ in the weak operator topology, in other words $\langle u, L_{g_n} v\rangle \to 0$ for all $u, v \in H$,
- $g \mapsto L_g$ is continuous in the strong operator topology, in other words $g \mapsto L_g u$ is continuous for all $u \in H$.

Theorem 5.1.3. *Let u_n be a bounded sequence. Up to extracting a subsequence, there exist* profiles $U^{(j)}$ *and* shifts $g_n^{(j)}$ *such that $g_n^{(j)}$ and $g_n^{(k)}$ are orthogonal for $j \neq k$ and*

$$u_n = \sum_{j=1}^{J} g_n^{(j)} U^{(j)} + r_n^{(J)}, \qquad \lim_{J\to\infty} \limsup_{n\to\infty} \|r_n^{(J)}\|_G = 0.$$

Moreover,

$$\|u_n\|_H^2 = \sum_{j=1}^{J} \|U^{(j)}\|_H^2 + \|r_n^{(J)}\|_H^2 + o(1), \qquad \text{as } n \to \infty.$$

The decomposition is essentially unique, meaning that $\|u_n\|_G \to 0$ *implies* $U^{(j)} = 0$ *for all* j.

Proof. □

Remark 5.1.4. The weak with concentration topology is the unique topology, for which the theorem above holds.

5.2 Description of the topology

5.2.1 Translations in $\mathbb{R}^d$

Let $H := H^1(\mathbb{R}^d)$ and $G = \mathbb{R}^d$ act by translations.

Theorem 5.2.1. *The topology of weak with concentration convergence on the unit ball is the* L^p *topology for any* $2 < p < 2^*$.

Proof. □

5.2.2 Translations and dilations in $\mathbb{R}^d$

Restrict to $d \geq 3$ and consider $(0,\infty) \times \mathbb{R}^d$ acting on $\dot H^s$ for some $s \in (0, d/2)$ by

$$(\lambda, x_0)u := x \mapsto \lambda^{\frac{d}{2}-s} u(\lambda(x - x_0)).$$

Theorem 5.2.2. *The topology of weak with concentration convergence on the unit ball is the* L^{2^*} *topology.*

Proof. Again, it is clear that the L^{2^*} topology is stronger than the G topology. We should show that if $\|u_n\|_{L^{2^*}} \geq 1$, then there exists a sequence g_n such that $g_n u_n$ does not converge weakly to 0. □

5.3 Translations, dilations and wave evolution

Finally, consider the group $\mathbb{R} \times (0,\infty) \times \mathbb{R}^d$ acting on $\dot H^1 \times L^2$ by

$$(t, \lambda, x_0)\boldsymbol{u}_0 := x \mapsto \boldsymbol{u}_L(t/\lambda, (x - x_0)/\lambda).$$

where $\boldsymbol{u}_L$ is the solution of (H) with initial data $\boldsymbol{u}_0$.

Theorem 5.3.1. *The weak with concentration topology is given by the Strichartz norms of* $\boldsymbol{u}_L$.

Proof. □

5.4 Nonlinear profile decomposition

For this part, I'm using my old notes written for the radial 3d power nonlinearity wave equation, available here: `https://math.univ-paris13.fr/~jendrej/other/M2.pdf`, pages 15–22. At some point I hope to write properly the equivariant wave maps version.

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