HIGHER EISENSTEIN ELEMENTS IN WEIGHT 2 AND PRIME LEVEL

In his classical work, Mazur considers the Eisenstein ideal I of the Hecke algebra $\mathbb{T}$ acting on cusp forms of weight 2 and level $\Gamma_0(N)$ where N is prime. When p is an Eisenstein prime, *i.e.* p divides the numerator of $\frac{N-1}{12}$, denote by $\mathbf{T}$ the completion of $\mathbb{T}$ at the maximal ideal generated by I and p. This is a $\mathbf{Z}_p$-algebra of finite rank $g_p \geq 1$ as a $\mathbf{Z}_p$-module.

Mazur asked what can be said about g_p. Merel was the first to study g_p. Assume for simplicity that $p \geq 5$. Let $\log : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{F}_p$ be a surjective morphism. Then Merel proved that

$$g_p \geq 2$$

if and only if

$$\sum_{k=1}^{\frac{N-1}{2}} k \cdot \log(k) \equiv 0 \text{ (modulo } p).$$

We prove that we have $g_p \geq 3$ if and only if

$$\sum_{k=1}^{\frac{N-1}{2}} k \cdot \log(k) \equiv \sum_{k=1}^{\frac{N-1}{2}} k \cdot \log(k)^2 \equiv 0 \text{ (modulo } p).$$

We also give a more complicated criterion to know when $g_p \geq 4$. Moreover, we prove *higher Eichler formulas.* More precisely, let

$$H(X) = \sum_{k=0}^{\frac{N-1}{2}} \binom{\frac{N-1}{2}}{k}^2 \cdot X^k \in \mathbf{F}_N[X]$$

be the classical Hasse polynomial. It is well-known that the roots of H are simple and in $\mathbf{F}_{N^2}^\times$. Let L be this set of roots. We prove that

$$\sum_{\lambda \in L} \log(H'(\lambda)) \equiv 4 \cdot \sum_{k=1}^{\frac{N-1}{2}} k \cdot \log(k) \text{ (modulo } p)$$

and, if $g_p \geq 2$,

$$\sum_{\lambda \in L} \log(H'(\lambda))^2 \equiv 4 \cdot \sum_{k=1}^{\frac{N-1}{2}} k \cdot \log(k)^2 \text{ (modulo } p) .$$