Solving Quadratic Integer Programs: Small Changes Yield Big Improvements

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Outline

Quadratic Constrained Quadratic Programming (QCQP)

(QCQP):

$$\min \ f(x) := x^T A x + 2a^T x \tag{1a}$$

$$\text{s.t.} \ h_i(x) := x^T B_i x + 2b_i^T x + c_i = (\leq) 0, \ \ i = 1, \ldots, m. \tag{1b}$$

Special cases: Binary Quadratic Program as for binary variables:

$$x_i \in \{0, 1\} \iff x_i^2 - x_i = 0.$$

NP-hard

Lagrangian Dual

Lagrange function:

$$L(x,\mu) = f(x) + \sum_i \mu_i h_i(x)$$
$$= x^T(A + \sum_i \mu_i B_i)x + 2(a + \sum_i \mu_i b_i)^T x + \sum_i \mu_i c_i,$$

where $\mu_i \geq 0$ for $h_i(x) \leq 0$.
Lagrangian dual problem of (QCQP) has an explicit formulation: Semidefinite programming (SDP):

$$\text{(D)} \qquad \sup_{\mu} \left\{ \inf_x L(x,\mu) \right\}$$
$$= \sup \sum_i \mu_i c_i - s$$
$$\text{s.t.} \begin{bmatrix} A + \sum_i \mu_i B_i & a + \sum_i \mu_i b_i \\ a^T + \sum_i \mu_i b_i^T & s \end{bmatrix} \succeq 0,$$

where $B \succeq 0$ stands for that B is positive semidefinite

Strong Duality for $m = 1$& inequality: S-Lemma

Let $f(x) = x^T Ax + 2a^T x + c$ and $h(x) = x^T Bx + 2b^T x + d$ be two quadratics having symmetric matrices A and B.
Under the Slater assumption, i.e., there is an $\overline{x} \in \mathbb{R}^n$ such that $h(\overline{x}) < 0$, the quadratic system

$$f(x) < 0, \ \ h(x) \leq 0 \tag{2}$$

is unsolvable if and only if there is a nonnegative number $\mu \geq 0$ such that

$$f(x) + \mu h(x) \geq 0, \ \ \forall x \in \mathbb{R}^n. \tag{3}$$

[1]Yakubovich, V.A.: S-procedure in nonlinear control theory. Vestnik Leningrad. Univ. 1, 62 – 77 (1971) (in Russian)
[2]Yakubovich, V.A.: S-procedure in nonlinear control theory. Vestnik Leningrad. Univ. 4, 73 – 93 (1977) (English translation)

S-Lemma with equality

Suppose Slater condition holds for $h(x) = 0$, i.e., there are x', x'' such that $h(x') < 0 < h(x'')$. S-Lemma with equality holds under one of the following additional assumptions:

(A) $h(x)$ is strictly concave (or convex), i.e., $B \prec (\succ) 0$.

(B) There is an $\eta \in \mathbb{R}$ such that $A \succeq \eta B$.

(C) $h(x)$ is homogeneous.

[A]Pólik, I., Terlaky, T. A survey of the S-lemma. SIAM Review, 49(3), 371-418 (2007)
[B]Beck, A., Eldar, Y.C.: Strong duality in nonconvex quadratic optimization with two quadratic constraint. SIAM J. OPTIM. 17(3), 844-860 (2006)
[C]Tuy, H., Tuan, H.D.: Generalized S-lemma and strong duality in nonconvex quadratic programming. J. Global Optim. 56(3):1045-1072 (2013)

S-lemma with equality: Our Result

Under the Slater Assumption that $h(x)$ takes both positive and negative values, the S-lemma with equality holds if h(x) is not linear, i.e., $B \neq 0$.

(Note that S-lemma with equality for the case $B = 0$ is easy to verify.)

S-lemma with equality $\Longrightarrow$ the classical S-lemma since $B \neq 0$ is satisfied when converting $h(x) \leq 0$ into $h(x) + t^2 = 0$.

[6] Y. Xia, S. Wang, R.L. Sheu, S-Lemma with Equality and Its Applications, arXiv:1403.2816v2 (2014) http://arxiv.org/abs/1403.2816

Generalized Trust-region subproblem

$$\begin{aligned} \min \quad & x^T A x + 2a^T x && \text{(4a)} \\ \text{s.t.} \quad & \alpha \leq x^T B x \leq \beta, && \text{(4b)} \end{aligned}$$

$$\begin{aligned} \text{(GTRS)} \quad \inf \quad & x^T A x + 2a^T x && \text{(5)} \\ \text{s.t.} \quad & \alpha \leq x^T B x + 2b^T x \leq \beta, && \text{(6)} \end{aligned}$$

[7] R.J. Stern and H. Wolkowicz, Indefinite trust region subproblems and nonsymmetric perturbations. SIAM J. Optim., 5(2), 286–313 (1995)

[8]Pong, T.K., Wolkowicz, H.: The generalized trust region subprobelm, Comput. Optim. Appl. 58, 273-322 (2014)

Pong and Wolkowicz's Result and Open Question

Pong and Wolkowicz have shown strong duality holds for (GTRS) under the following assumption:

Assumption

1. $B \neq 0$.
2. *(GTRS) is feasible.*
3. *The following relative interior constraint qualification holds*

$$\text{(RICQ)}\ \alpha < \mathrm{tr}(B\widehat{X}) + 2b^T\widehat{x} + d < \beta, \ \text{for some}\ \widehat{X} \succ \widehat{x}\widehat{x}^T.$$

4. *(GTRS) is bounded below.*
5. *The dual of (GTRS) is feasible.*

Under Assumptions 1,2,3, it is trivial to see Item 5 $\Longrightarrow$ Item 4. They have proved when $b = 0$, Item 4 $\Longrightarrow$ Item 5. An open question was raised when $b \neq 0$.

S-lemma with interval bounds

Under the Slater Assumption that there exists an $\overline{x} \in \mathbb{R}^n$ such that $\alpha < h(\overline{x}) < \beta$, S-lemma with interval bounds holds when $B \neq 0$, i.e., the system $f(x) < 0,\ \alpha \leq h(x) \leq \beta$ is unsolvable if and only if there is a number $\mu \in \mathbb{R}$ such that

$$f(x) + \mu_-(h(x) - \beta) + \mu_+(\alpha - h(x)) \geq 0,\ \forall x \in \mathbb{R}^n.$$

where $\mu_+ = \max\{\mu, 0\}, \mu_- = -\min\{\mu, 0\}$.

Corollary

Under Items $1, 2, 3$ *in Pong and Wolkowicz's Assumption, strong duality holds for* (GTRS). *Moreover, under Items* $1, 2, 3$ *in Pong and Wolkowicz's Assumption, Items 4 and 5 are equivalent.*

[9]Shu Wang, Yong Xia, Strong Duality for Generalized Trust Region Subproblem: S-Lemma with Interval Bounds, 2014 working paper

Approximate Algorithms: an example

Rather than providing relaxations, SDP also has applications in giving approximate algorithms. For example,

$$\begin{array}{llll} \text{(ECQP)} & \min_{x\in\mathbb{R}^n} & f(x) = x^T A x + 2b^T x & (7) \\ & \text{s.t.} & \|F^k x + g^k\|_2^2 \le 1,\ k = 1,\ldots,m, & (8) \end{array}$$

where $\|g_k\| < 1$ is assumed.

The semidefinite programming relaxation of (ECQP) is

$$\begin{array}{lll} \text{(SDP)} & \min & B \bullet X \\ & \text{s.t.} & B^k \bullet X \le 0,\ k = 1,\ldots,m, \\ & & X_{n+1,n+1} = 1,\ X \succeq 0,\ X \in \mathbb{R}^{(n+1)\times(n+1)}. \end{array}$$

Approximate Algorithms: an example

Theorem (Tseng 2003)

For (ECQP), we can generate a feasible solution in polynomial time satisfying

$$f(x) \leq \frac{(1-\gamma)^2}{(\sqrt{m}+\gamma)^2} \cdot v(\mathrm{SDP}), \qquad (9)$$

where $\gamma := \max_{k=1,\ldots,m} \|g^k\|$.

Very recently, we can show that the m in (9) can be improved to

$$\min\left\{\left\lceil \frac{\sqrt{8m+17}-3}{2} \right\rceil, n+1\right\}.$$

[10]P. Tseng, Further results on approximating nonconvex quadratic optimization by semidefinite programming relaxation, SIAM Journal Optimization, 14, 2003, 268-283

Summary of Applications of SDP

- Providing efficient relaxations
- Strong duality for special QCQP
- Establishing approximate algorithms
- Providing high-quality reformulations (The remaining of this talk)

Quadratic Convex Reformulation (QCR)

$$\begin{array}{lll} \text{(P)} & \min & x^T Qx + c^T x \\ & \text{s.t.} & x \in \{0,1\}^n. \end{array}$$

Note that for $x \in \{0,1\}^n$, we always have

$$x^T Qx + c^T x = x^T Qx + c^T x + \sum_{i=1}^{n} \theta_i x_i (x_i - 1), \ \forall \theta \in \mathbb{R}^n.$$

Thus, (P) is equivalent to

$$\begin{array}{lll} (\text{P}_\theta) & \min & x^T (Q + \text{Diag}(\theta))x + (c - \theta)^T x \\ & \text{s.t.} & x \in \{0,1\}^n. \end{array}$$

What is the "best" choice of θ?

Quadratic Convex Reformulation (QCR)

A good choice of θ can be obtained by maximizing the continuous relaxation

$$\sup_{\theta}\left\{\inf_{x}\left\{x^T(Q+\mathrm{Diag}(\theta))x+(c-\theta)^Tx\right\}\right\},$$

which can be reformulated as an SDP:

$$\theta^* = arg \max_{\begin{bmatrix} 2Q+2\mathrm{Diag}(\theta) & c-\theta \\ c^T-\theta^T & -s \end{bmatrix}\succeq 0} s$$

[11]A. Billionnet, S. Elloumi: Using a mixed integer quadratic programming solver for the unconstrained quadratic 0-1 problem. Mathematical Programming. 109, 55-68 (2007)
[12]A. Billionnet, S. Elloumi, M.-C. Plateau: Improving the performance of standard solvers for quadratic 0-1 programs by a tight convex reformulation: the QCR method. Discrete Applied Mathematics. 157(6) 1185-1197 (2009)

On Extensions of QCR

Limits of QCR:

- QCR works well for equality constraints. However, equivalence fails for inequality constraints.
- QCR works well for binary variables. How about general integer variables?

These will be discussed in two cases:

- Probabilistically constrained quadratic programs
- Box-constrained nonconvex quadratic integer program

Probabilistically Constrained Quadratic Programs

$$
\begin{aligned}
(\mathrm{P}) \qquad \min \quad & x^T Q x + c^T x \\
\text{s.t.} \quad & \mathbb{P}(\xi^T B x \geq R) \geq 1 - \epsilon, \\
& x \in X,
\end{aligned}
$$

where $Q \in \Re^{n \times n}$ is positive semi-definite, $c \in \Re^n$, $\mathbb{P}$ denotes the probability, ξ is an m-dimensional random vector that takes finitely many realizations (scenarios), $\xi^1, \ldots, \xi^N \in \Re^m$, with equal probability, B is an $m \times n$ matrix, $R \in \Re$, $1 - \epsilon \in (0, 1)$ is the confidence level, X is assumed to be a bounded convex set. When $Q = 0$, Problem (P) is already NP-hard.

Reformulation: Mixed-integer quadratic program

$$
\begin{aligned}
(\mathrm{MIQP}_0) \quad & \min \quad x^T Q x + c^T x \\
& \text{s.t.} \quad (\xi^i)^T B x \geq R + y_i(\alpha_i - R), \ i = 1, \ldots, N, \\
& \qquad e^T y \leq K, \ y \in \{0,1\}^N, \\
& \qquad x \in X,
\end{aligned}
$$

where $\alpha_i = \min_{x \in X} \ (\xi^i)^T Bx, \ \beta_i = \max_{x \in X} \ (\xi^i)^T Bx$.

[13]Benati, S., Rizzi, R., 2007. A mixed integer linear programming formulation of the optimal mean/Value-at- Risk portfolio problem, European Journal of Operational Research 176, 423-434.

[14]Ruszczyński, A., Probabilistic programming with discrete distributions and precedence constrained knapsack polyhedra. *Math. Program.* 93, 195–215 (2002)

QCR based on Lagrangian decomposition

Recently, based on a Lagrangian decomposition approach, Zheng et al. develop the following class of reformulations with a parameter θ, denoted by (MIQP_θ):

$$
\begin{aligned}
\min \quad & x^T(Q - \sum_{i=1}^{N} \theta_i B^T \xi^i (\xi^i)^T B)x + \sum_{i=1}^{N} \theta_i (w_i^2 + \phi_i - R^2 y_i) + c \\
\text{s.t.} \quad & (\xi^i)^T B x = w_i + z_i - y_i R, \ i = 1, \ldots, N, \\
& e^T y \le K, \ y \in \{0,1\}^N, \\
& \alpha_i y_i \le z_i \le R y_i, \ \phi_i y_i \ge z_i^2, \ \phi_i \ge 0, \ i = 1, \ldots, N, \\
& x \in X, \ R \le w_i \le \beta_i, \ i = 1, \ldots, N.
\end{aligned}
$$

[15]Zheng, X.J., Sun, X.L., Li, D. and Cui, X.T. Lagrangian Decomposition and Mixed-Integer Quadratic Programming Reformulations for Probabilistically Constrained Quadratic Programs, European Journal of Operational Research, 221, 38-48 (2012)

Let $v(\cdot)$ and $v^c(\cdot)$ be the optimal values of problem $(\cdot)$ and its continuous relaxation, respectively. Define

$$\Theta = \{\theta \in \Re^N \mid Q - \sum_{i=1}^{N} \theta_i B^T \xi^i (\xi^i)^T B \succeq 0, \ \theta \geq 0\}, \qquad (10)$$

For any $\theta \in \Theta$, while $v(\mathrm{MIQP}_0) = v(\mathrm{MIQP}_\theta)$ $v^c(\mathrm{MIQP}_\theta) \geq v^c(\mathrm{MIQP}_0)$, implying that reformulation (MIQP_θ) is more efficient than reformulation (MIQP_0) as the continuous relaxation of (MIQP_θ) always generates a lower bound tighter than or at least as tight as that of (MIQP_0).

Three Choice of θ

The first choice of θ is:

$$\theta^* = arg \max_{\theta \in \Theta} \{\mathrm{SDP}(\theta)\}, \tag{11}$$

where $(\mathrm{SDP}(\theta))$ is the conic dual of the continuous relaxation of (MIQP_θ). The other two heuristic choices of θ are based on approximating (SDP). The second one is

$$\theta^s = arg \max_{\theta \in \Theta} e^T \theta. \tag{12}$$

When Q is positive definite, the third choice of θ is

$$\theta^e = \frac{1}{\lambda_{\max}(U^T B^T \Gamma \Gamma^T B U)} e, \tag{13}$$

where U is the orthogonal matrix such that $U^T Q U = I$, $\lambda_{\max}(C)$ denotes the maximum eigenvalue of C and $\Gamma = (\xi^1, \ldots, \xi^N)$.

Among the three reformulations, $(\mathrm{MIQP}_{\theta^*})$ is the most efficient according to numerical results.

Observation

$$
\begin{aligned}
\min \quad & x^T(Q - \sum_{i=1}^N \theta_i B^T \xi^i (\xi^i)^T B)x + \sum_{i=1}^N \theta_i (w_i^2 + \phi_i - R^2 y_i) + c \\
\text{s.t.} \quad & (\xi^i)^T Bx = w_i + z_i - y_i R, \ i = 1, \ldots, N, \\
& e^T y \le K, \ y \in \{0,1\}^N, \\
& \alpha_i y_i \le z_i \le R y_i, \ \phi_i y_i \ge z_i^2, \ \phi_i \ge 0, \ i = 1, \ldots, N, \\
& x \in X, \ R \le w_i \le \beta_i, \ i = 1, \ldots, N.
\end{aligned}
$$

If $y_i = 0$, then $z_i = 0$ and $\phi_i = 0$.
If $y_i = 1$, then $\phi_i = z_i^2$.

QCR: New reformulation

$(\mathrm{MIQP}_\theta^{\mathrm{new}})$:

$$\begin{array}{ll} \min & x^T(Q - \sum_{i=1}^N \theta_i B^T \xi^i (\xi^i)^T B)x + \sum_{i=1}^N \theta_i (w_i^2 + z_i^2 - R^2 y_i) + \\ \mathrm{s.t.} & (\xi^i)^T Bx = w_i + z_i - y_i R,\ i = 1, \ldots, N, \\ & e^T y \le K,\ y \in \{0,1\}^N, \\ & \alpha_i y_i \le z_i \le R y_i,\ i = 1, \ldots, N, \\ & x \in X,\ R \le w_i,\ i = 1, \ldots, N. \end{array}$$

The relaxation of $(\mathrm{MIQP}_\theta^{\mathrm{new}})$ is a convex quadratic program while that of (MIQP_θ) is a second-order constrained quadratic program.

New explanation

For the inequality constraint

$$(\xi^i)^T Bx \geq R + y_i(\alpha_i - R)$$

Introducing

$$\begin{aligned} w_i &= \max\left\{(\xi^i)^T Bx, R\right\}, \\ z_i &= \min\left\{(\xi^i)^T Bx, R\right\} - R + y_i R, \end{aligned}$$

then we have not only

$$(\xi^i)^T Bx = w_i + z_i - y_i R$$

but also

$$\left((\xi^i)^T Bx\right)^2 = w_i^2 + z_i^2 - R^2 y_i.$$

The Choice of θ

Theorem

For any $\theta \in \Theta$,

$$v^c(\mathrm{MIQP}_\theta^{\mathrm{new}}) - v^c(\mathrm{MIQP}_0) \geq -\frac{K(N-K)}{N} R^2 \left(\max_i \theta_i\right). \quad (14)$$

The "best" choice of θ:

$$\theta^* = arg \max_{\theta \in \Theta} \{v^c(\mathrm{MIQP}_\theta^{\mathrm{new}})\}.$$

Theorem

$$v^c(\mathrm{MIQP}_{\theta^*}^{\mathrm{new}}) \geq v^c(\mathrm{MIQP}_0).$$

Numerical results for the VaR-variance problems

Numerical tests were implemented in Matlab R2011a and IBM ILOG CPLEX 12.3 and run on a 64bit Linux (3GHz, 8GB RAM).

The maximum CPU time limit is set to 3600 seconds for $n \leq 150$ and 7200 seconds for $n > 150$.

Computational results with ε=0.05

	$(MIQP_0)$		$(MIQP_{\theta^*})$		$(MIQP_{\theta^*}^{\mathrm{new}})$	
n	**Time**	**Nodes**	**Time**	**Nodes**	**Time**	**Nodes**
120	2130.74	1461	261.20	330	7.55	613
120	3600.02	2240	708.25	931	39.39	4916
120	3600.02	2520	243.11	309	21.40	2937
120	3600.02	2100	411.83	592	18.47	1537
120	3600.01	2279	1944.14	2885	67.16	14163
150	3600.02	1280	3600.00	3770	2080.15	232633
150	3600.01	1372	3600.00	4090	410.99	37618
150	3600.04	1184	3600.01	3120	432.28	59478
150	3600.01	1200	3600.01	4070	3600.01	359194
150	3600.03	1240	3600.00	3168	447.63	51829
175	509.99	77	310.73	136	15.54	68
175	7200.02	1242	3773.94	1558	164.64	5594
175	7200.01	977	7200.01	2700	3251.24	147063
175	534.83	65	1072.88	509	16.64	53
175	7200.02	1371	4132.90	2963	225.36	9499
175	7200.02	1032	7200.01	3547	82.66	3720
200	7200.04	889	7200.02	2766	2702.59	120183
200	2479.95	256	273.53	127	23.88	210

Computational results with ε=0.1

	$(MIQP_0)$		$(MIQP_{\theta*})$		$(MIQP_{\theta*}^{\text{new}})$	
n	**Time**	**Nodes**	**Time**	**Nodes**	**Time**	**Nodes**
100	3600.00	3418	1819.72	3930	38.29	12630
100	3600.02	3893	558.44	1192	24.75	6432
100	3600.01	3500	1713.26	3532	35.57	9136
100	3600.00	3946	985.90	1907	56.55	12967
100	3600.01	4129	1705.62	3649	50.83	14335
120	3600.01	1280	1259.78	1872	58.13	13740
120	3600.01	1372	3600.01	4773	3600.00	895540
120	3600.01	1184	3600.00	5618	100.11	19664
120	3600.02	1200	3600.01	5245	175.53	34633
120	3600.02	1240	3600.02	5543	744.08	145174
150	3600.01	1237	3600.00	3265	292.26	39990
150	3600.00	1425	3600.01	3086	1759.60	267985
150	3600.00	1585	3600.01	4123	682.76	85367
150	3600.01	1228	3600.00	3936	3600.01	507000
150	3600.00	1151	2772.83	2564	1279.55	176624

Computational results with ε=0.1

	$(MIQP_0)$		$(MIQP_{\theta^*})$		$(MIQP_{\theta^*}^{\text{new}})$	
n	**Time**	**Nodes**	**Time**	**Nodes**	**Time**	**Nodes**
175	7200.01	1248	7200.01	3248	2446.51	130749
175	2294.58	427	3217.28	2433	18.05	292
175	7200.02	1200	7200.01	2664	7200.00	406668
175	7200.01	1148	7200.01	3921	478.67	19300
175	7200.04	1426	6442.23	3672	495.73	17821
175	7200.01	1204	7200.01	4171	704.87	31840
200	7200.02	821	7200.02	754	7200.03	200128
200	7200.01	726	7200.04	741	7200.02	188436
200	7200.02	820	7200.03	746	3777.10	159977
200	1563.94	165	1932.86	220	22.28	244
200	7200.03	600	7200.05	600	7200.03	268871
200	7200.04	762	7200.03	789	409.41	8710

Quadratic integer program

Box-constrained nonconvex quadratic integer program:

$$
\begin{aligned}
\text{(P)} \quad \min \quad & x^T Qx + c^T x \\
\text{s.t.} \quad & l_i \le x_i \le u_i, \;\; i = 1, \ldots, n, \\
& x \in Z^n,
\end{aligned}
$$

where Z^n denotes the set of n-dimensional vectors with integer entries, $l, u \in Z^n$ and $l_i < u_i$ for $i = 1, \ldots, n$.

Solvers

CPLEX 12.6 can solve (P) with $Q \not\succeq 0$.

Q-MIST: efficient branch-and-bound approaches based on SDP relaxation

GQIP: efficient branch-and-bound approaches based on ellipsoidal relaxation

[16]C. Buchheim, M. D. Santis, L. Palagi, M. Piacentini, An Exact Algorithm for Nonconvex Quadratic Integer Minimization using Ellipsoidal Relaxations, SIAM Journal on Optimization 23(3) (2013) 1867–1889

[17]C. Buchheim and A. Wiegele, Semidefinite relaxations for non-convex quadratic mixed-integer programming. Math. Program., 141(1-2) (2013) 435–452

Quadratic Convex Reformulation and extension

Assume $l_i = 0$ for $i = 1, \ldots, n$. If $u_i = 1$ for $i = 1, \ldots, n$, the nonconvex objective of problem (P) is easy to convexify:

$$x^T Qx + c^T x = x^T Qx + c^T x + \sum_{i=1}^{n} \sigma_i x_i (x_i - 1), \ \forall \sigma \in \mathbb{R}^n.$$

Generally,

$$x^T Qx + c^T x = x^T Qx + c^T x + \sum_{i=1}^{n} \sigma_i \left(x_i^2 - \sum_{k=0}^{u_i} k^2 y_{ik} \right),$$

where y_{ik} are additional binary variables satisfying

$$\sum_{k=0}^{u_i} y_{ik} = 1, \ x_i = \sum_{k=0}^{u_i} k y_{ik}. \tag{15}$$

[18]A. Billionnet, S. Elloumi, A. Lambert. Extending the QCR method to general mixed integer programs. Math. Program. 131(1) (2012) 381–401

QCR

The above approach was called (NC) (Naive Convexification) when setting $\sigma_i = -\lambda_{\min}(Q)$ for $i = 1, \ldots, n$.
Here we call it $(\mathrm{NC}_{\mathrm{MinEig}})$ as we have the other version of NC, denoted by $(\mathrm{NC}_{\mathrm{SDP}})$, where σ is the "best" choice obtained by maximizing the corresponding SDP relaxation.

Another reformulation approach: introduce the unique binary decomposition

$$x_i = \sum_{k=0}^{\lfloor \log(u_i) \rfloor} 2^k t_{ik}$$

and new variables $y_{ij} := x_i x_j$ and $z_{ijk} := t_{ik} x_j$ to linearize the equalities:

$$x_i x_j = \sum_{k=0}^{\lfloor \log(u_i) \rfloor} 2^k t_{ik} x_j.$$

Our Approach: Step 1

Rewrite (P) as

$$(\widetilde{\mathrm{P}}) \quad \min \ \widetilde{x}^T Q \widetilde{x} + c^T \widetilde{x}$$
$$\text{s.t.} \quad -m_i \le \widetilde{x}_i \le m_i, \ \widetilde{x}_i \in Z, \ \forall i \in I,$$
$$-m_i \le \widetilde{x}_i \le m_i, \ \frac{\widetilde{x}_i - 1}{2} \in Z, \ \forall i \in J,$$

where

$$I = \left\{ i : \ \frac{u_i + l_i}{2} \in Z \right\}, \ J = \left\{ i : \ \frac{u_i + l_i}{2} \notin Z \right\},$$

$$m_i = \begin{cases} \frac{u_i - l_i}{2}, & \text{if } i \in I \\ u_i - l_i, & \text{if } i \in J \end{cases}, \ \widetilde{x}_i = \begin{cases} x_i - \frac{u_i + l_i}{2}, & \text{if } i \in I, \\ 2x_i - (u_i + l_i), & \text{if } i \in J. \end{cases}$$

Here, for simplicity, we assume $J = \emptyset$.

Our Approach: Step 2

$(\widetilde{\mathrm{P}})$ is equivalent to

$$
\begin{aligned}
(\mathrm{MBQP}_\theta) \quad \min \quad & \widetilde{x}^T Q \widetilde{x} + c^T \widetilde{x} + \sum_{i \in I} \theta_i \left(\widetilde{x}_i^2 - \sum_{k=1}^{m_i} k^2 y_{ik} \right) \\
\text{s.t.} \quad & -\sum_{k=1}^{m_i} k y_{ik} \le \widetilde{x}_i \le \sum_{k=1}^{m_i} k y_{ik}, \ \forall i \in I, \\
& z_i \le \sum_{k=1}^{m_i} y_{ik} \le 1, \ \forall i \in I, \\
& \sum_{k=1}^{m_i} k y_{ik} - \widetilde{x}_i \le 2 m_i z_i, \ \forall i \in I, \\
& \sum_{k=1}^{m_i} k y_{ik} + \widetilde{x}_i \le 2 m_i (1 - z_i), \ \forall i \in I, \\
& y_{ik} \in \{0,1\}, \forall i, k; \ z \in \{0,1\}^n.
\end{aligned}
$$

Benefit

The naive representation of $\{x_i : \ l_i \le x_i \le u_i, \ x_i \in Z\}$

$$\left\{ x_i = \sum_{k=l_i}^{u_i} k y_{ik} : \ \sum_{k=l_i}^{u_i} y_{ik} = 1, \ y_{ik} \in \{0,1\}^{u_i - l_i + 1} \right\}.$$

needs $u_i - l_i + 1$ additional binary variables y_k.

In our reformulation (MBQP_θ), we introduce an additional binary variable z_i and $\frac{u_i - l_i}{2}$ additional binary variables y_{ik}.

The choice of θ

For any θ such that $Q + \mathrm{Diag}(\theta) \succeq 0$, (MBQP_θ) has a convex objective function. A "best" choice of θ, denoted by θ^*, seems to be the one that maximizes the continuous relaxation of (MBQP_θ). For convenience, we rewrite the continuous relaxation of (MBQP_θ) as

$$\begin{array}{lll} \mathrm{R}(\theta) & \min & \tilde{x}^T(Q + \mathrm{Diag}(\theta))\tilde{x} + c^T\tilde{x} - L(\theta)^T y \\ & \mathrm{s.t.} & A\tilde{x} + By \le a, \end{array}$$

Since $Q + \mathrm{Diag}(\theta) \succeq 0$, by strong duality, we have

$$v(\mathrm{R}(\theta)) = \max_{\lambda \ge 0} \ \tau - a^T\lambda$$
$$\mathrm{s.t.} \ \begin{bmatrix} -\tau & \frac{1}{2}(c + A^T\lambda)^T & \frac{1}{2}(-L(\theta) + B^T\lambda)^T \\ \frac{1}{2}(c + A^T\lambda) & Q + \mathrm{Diag}(\theta) & 0 \\ \frac{1}{2}(-L(\theta) + B^T\lambda) & 0 & 0 \end{bmatrix} \succeq 0. \quad (16)$$

θ^* is obtained by solving an SDP:

$$\theta^* = \arg \max_{Q + \mathrm{Diag}(\theta) \succeq 0} v(\mathrm{R}(\theta)) = \max_{\lambda \ge 0, \ (16)} \left\{ \tau - a^T\lambda \right\}, \quad (17)$$

Experimental Results for instances with variable domain $\{-5, \ldots, 5\}$

n	ALG	SOLVED	MAX TIME	AVG TIME	AVG #N
20	P	10	2.0	0.5	1637.6
	NC_{SDP}	110	55. 7	0.7	2435.9
	$\text{NC}_{\text{MinEig}}$	110	1131.2	20.4	76705.4
	Q-MIST	110	6.0	0.8	138.6
	GQIP	110	56.9	1.9	1568914
	MBQP_{θ^*}	110	1.0	0.1	191. 5
30	P	10	30.7	11.8	33761.6
	NC_{SDP}	108	1171.8	13.7	32681.4
	$\text{NC}_{\text{MinEig}}$	100	1037.2	78.1	213661.
	Q-MIST	110	237.0	17.8	1115.5
	GQIP	103	3175.1	256.6	1518893
	MBQP_{θ^*}	110	8.2	1.0	1746.9

Experimental Results for instances with variable domain $\{-5, \ldots, 5\}$

n	ALG	SOLVED	MAX TIME	AVG TIME	AVG #N
40	P	10	1446.1	354.9	680782.6
	$\mathrm{NC}_{\mathrm{MinEig}}$	76	1020.4	79.4	166358.7
	Q-MIST	109	2431.0	211.3	5861.5
	GQIP	32	3501.8	600.6	2547514
	MBQP_{θ^*}	110	229.7	12.2	21478.8

Thanks

Thank you for your time!