l_p-Norm Constrained Quadratic Programming: Conic Approximation Methods

Wenxun Xing
Department of Mathematical Sciences
Tsinghua University, Beijing
Email: wxing@math.tsinghua.edu.cn

OUTLINE

1. l_p-Norm Constrained Quadratic Programming
2. Linear Conic Programming Reformulation
3. Complexity
4. Approximation Scheme
5. Questions

Data fitting

- l_2-norm: least-square data fitting

$$\begin{array}{ll} \min & \|Ax-b\|_2 \\ \text{s.t.} & x \in \mathbb{R}^n. \end{array}$$

- When A is full rank in column, then $x^* = (A^TA)^{-1}A^Tb$.
- A 2nd-order conic programming formulation

$$\begin{array}{ll} \min & t \\ \text{s.t.} & \|Ax-b\|_2 \leq t \\ & x \in \mathbb{R}^n. \end{array}$$

- Experts in numerical analysis prefer the direct calculation much more than the optimal solution method.

l_1- norm problem

- l_1-norm.

$$
\begin{aligned}
\min \quad & \|x\|_1 \\
\text{s.t.} \quad & Ax = b \\
& x \in \mathbb{R}^n.
\end{aligned}
$$

- A linear programming formulation

$$
\begin{aligned}
\min \quad & \textstyle\sum_{i=1}^n t_i \\
\text{s.t.} \quad & -t_i \le x_i \le t_i,\ i = 1, 2, \ldots, n \\
& Ax = b \\
& t, x \in \mathbb{R}^n.
\end{aligned}
$$

Heuristic method for finding a sparse solution

- Regressor selection problem: A potential regressors, b to be fit by a linear combination of A

$$\begin{aligned} \min \quad & \|Ax-b\|_2 \\ \text{s.t.} \quad & card(x) \le k \\ & x \in \mathbb{Z}_+^n. \end{aligned}$$

- It is NP-hard. Let $m=1$, $A=(a_1,a_2,\ldots,a_n)$, $b=\frac{1}{2}\sum_{i=1}^n a_i$, $k \le \frac{n}{2}$. It is a partition problem.
- Heuristic method.

$$\begin{aligned} \min \quad & \|Ax-b\|_2 + \gamma\|x\|_1 \\ \text{s.t.} \quad & x \in \mathbb{R}^n. \end{aligned}$$

- Ref. S. Boyd and L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.

Regularized approximation

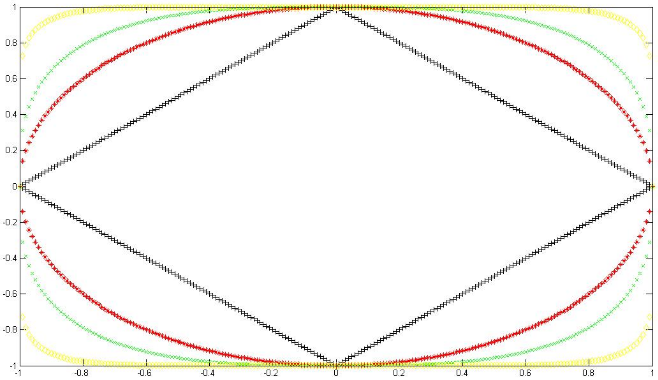
$$\begin{aligned} \min \quad & \|Ax - b\|_2 + \gamma \|x\|_1 \\ \text{s.t.} \quad & x \in \mathbb{R}^n. \end{aligned}$$

- l_1-norm and l_2-norm constrained programming

$$\begin{aligned} \min \quad & t_1 + \gamma t_2 \\ \text{s.t.} \quad & \|Ax - b\|_2 \leq t_1 \\ & \|x\|_1 \leq t_2 \\ & x \in \mathbb{R}^n, t_1, t_2 \in \mathbb{R}. \end{aligned}$$

- The objective function is linear, the first constraint is a 2nd-order cone and the 2nd is a 1st-order cone.
- It is a convex optimization problem of polynomially solvable.

p-norm domain



Black: 1-norm. Red: 2-norm. Green: 3-norm. Yellow: 8-norm.

Convex l_p-norm problems

- p-norm domain is convex ($p \geq 1$).
- For set $\{x \mid \|x\|_p \leq 1\}$, the smallest one is the domain with $p = 1$, which is the smallest convex set containing integer points $\{-1, 1\}^n$.
- For $p \geq 1$, the l_p-norm problems with linear objective or linear constraints are polynomially solvable.
- Variants of l_p-norm problems should be considered.

Variants of l_p-norm problems

- l_2-norm constrained quadratic problem

$$\begin{aligned} \min \quad & x^T Qx + q^T x \\ \text{s.t.} \quad & \|Ax - b\|_2 \le c^T x \\ & c^T x = d \ge 0 \\ & x \in \mathbb{R}^n. \end{aligned}$$

- l_1-norm constrained quadratic problem

$$\begin{aligned} \min \quad & x^T Qx + q^T x \\ \text{s.t.} \quad & \|x\|_1 \le k \\ & x \in \mathbb{R}^n, \end{aligned}$$

where Q is a general symmetric matrix.

l_p-Norm Constrained Quadratic Programming

$$
\begin{aligned}
\min \quad & \tfrac{1}{2}x^TQx + q^Tx \\
\text{s.t.} \quad & \tfrac{1}{2}x^TQ_ix + q_i^Tx + c_i \le 0, i = 1, 2, \ldots, m \\
& \|Ax - b\|_p \le c^Tx \\
& x \in \mathbb{R}^n,
\end{aligned}
$$

where $p \ge 1$.

QCQP reformulation

$$\begin{array}{rl} \min & \frac{1}{2}x^T Q_0 x + q_0^T x + c_0 \\ \text{s.t.} & \frac{1}{2}x^T Q_i x + q_i^T x + c_i \leq 0, i = 1, 2, \ldots, m \\ & x \in \mathcal{D}, \end{array}$$

where $\mathcal{D} \subseteq \mathbb{R}^n$.

p-norm form

- l_1-norm problem

$$\begin{aligned} \min \quad & x^T Qx + q^T x \\ \text{s.t.} \quad & \|x\|_1 \le k \\ & x \in \mathbb{R}^n. \end{aligned}$$

Denote $\mathcal{D} = \{x \in \mathbb{R}^n \mid \|x\|_1 \le k\}$.

- QCQP form

$$\begin{aligned} \min \quad & x^T Qx + q^T x \\ \text{s.t.} \quad & x \in \mathcal{D}. \end{aligned}$$

2-norm form

- 2-norm problem

$$\begin{aligned} \min \quad & x^T Qx + q^T x \\ \text{s.t.} \quad & \|Ax - b\|_2 \leq c^T x \\ & c^T x = d \geq 0 \\ & x \in \mathbb{R}^n. \end{aligned}$$

Denote $\mathcal{D} = \left\{x \in \mathbb{R}^n \mid \|Ax - b\|_2 \leq c^T x\right\}$

- QCQP form

$$\begin{aligned} \min \quad & x^T Qx + q^T x \\ \text{s.t.} \quad & c^T x = d \\ & x \in \mathcal{D}. \end{aligned}$$

Lifting reformulation

$$
\begin{aligned}
\min \quad & f(x) = \tfrac{1}{2}x^T Q_0 x + q_0^T x + c_0 \\
\text{s.t.} \quad & g_i(x) = \tfrac{1}{2}x^T Q_i x + q_i^T x + c_i \leq 0, i = 1, 2, \ldots, m \quad (\color{red}{QCQP}) \\
& x \in \mathcal{D}.
\end{aligned}
$$

Denote: $\mathcal{F} = \{x \in \mathcal{D} \mid g_i(x) \leq 0, i = 1, 2, \ldots, m\}$.

- Lifting

$$
\begin{aligned}
\min \quad & \frac{1}{2}\begin{pmatrix} 2c_0 & q_0^T \\ q_0 & Q_0 \end{pmatrix} \bullet X \\
\text{s.t.} \quad & \frac{1}{2}\begin{pmatrix} 2c_i & q_i^T \\ q_i & Q_i \end{pmatrix} \bullet X \leq 0, i = 1, 2, \ldots, m \\
& X = \begin{pmatrix} 1 \\ x \end{pmatrix}\begin{pmatrix} 1 \\ x \end{pmatrix}^T, \quad x \in \mathcal{F}.
\end{aligned}
$$

Convex reformulation

$$\begin{array}{rl} \min & \frac{1}{2}\begin{pmatrix} 2c_0 & q_0^T \\ q_0 & Q_0 \end{pmatrix} \bullet X \\ \text{s.t.} & \frac{1}{2}\begin{pmatrix} 2c_i & q_i^T \\ q_i & Q_i \end{pmatrix} \bullet X \le 0, i = 1, 2, \ldots, m \\ & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet X = 1 \\ & X \in \mathrm{cl}(\mathrm{conv}(\left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T | x \in \mathcal{F} \right\})). \end{array}$$

Linear conic programming reformulation

$$
\begin{aligned}
\min \quad & \frac{1}{2}\begin{pmatrix} 2c_0 & q_0^T \\ q_0 & Q_0 \end{pmatrix} \bullet X \\
\text{s.t.} \quad & \frac{1}{2}\begin{pmatrix} 2c_i & q_i^T \\ q_i & Q_i \end{pmatrix} \bullet X \le 0, i = 1, 2, \ldots, m \\
& \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet X = 1 \\
& X \in \mathrm{cl}(\mathrm{cone}(\left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T | x \in \mathcal{F} \right\})).
\end{aligned}
$$

- It is a linear conic programming and has the same optimal value with QCQP.

Quadratic-Function Conic Programming

- PRIMAL

$$\begin{aligned} \min \quad & \frac{1}{2}\begin{pmatrix} 2c_0 & q_0^T \\ q_0 & Q_0 \end{pmatrix} \bullet V \\ \text{s.t.} \quad & \frac{1}{2}\begin{pmatrix} 2c_i & q_i^T \\ q_i & Q_i \end{pmatrix} \bullet V \leq 0, i = 1, 2, \ldots, m \qquad (QFCP) \\ & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet V = 1 \\ & V \in \mathcal{D}_{\mathcal{F}}^* = \text{cl}\left(\text{cone}\left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T, x \in \mathcal{F} \right\} \right). \end{aligned}$$

- $\mathcal{F} \subseteq \mathbb{R}^n$, $A \bullet B = trace(AB^T)$,

Quadratic-Function Conic Programming

- DUAL

$$
\begin{aligned}
&\max\ \sigma \\
\text{s.t.}\ &\begin{pmatrix} -2\sigma + 2c_0 + 2\sum_{i=1}^m \lambda_i c_i & (q_0 + \sum_{i=1}^m \lambda_i q_i)^T \\ q_0 + \sum_{i=1}^m \lambda_i q_i & Q_0 + \sum_{i=1}^m \lambda_i Q_i \end{pmatrix} \in \mathcal{D}_{\mathcal{F}} \\
&\sigma \in \mathbb{R}, \lambda \in \mathbb{R}^m_+,
\end{aligned}
$$

$\mathcal{F} \subseteq \mathbb{R}^n$,

$$
\mathcal{D}_{\mathcal{F}} = \left\{ U \in \mathcal{S}^{n+1} \middle| \begin{pmatrix} 1 \\ x \end{pmatrix}^T U \begin{pmatrix} 1 \\ x \end{pmatrix} \geq 0, \forall\, x \in \mathcal{F} \right\}.
$$

Properties of the Quadratic-Function Cone

- Cone of nonnegative quadratic functions (Sturm and Zhang, MOR 28, 2003).

$$\mathcal{D}_{\mathcal{F}} = \left\{ U \in \mathcal{S}^{n+1} \middle| \begin{pmatrix} 1 \\ x \end{pmatrix}^T U \begin{pmatrix} 1 \\ x \end{pmatrix} \geq 0, \forall\, x \in \mathcal{F} \right\}.$$

- If $\mathcal{F} \neq \emptyset$, then $\mathcal{D}_{\mathcal{F}}^*$ is the dual cone of $\mathcal{D}_{\mathcal{F}}$ and vice versa.
- If $\mathcal{F}$ is a bounded nonempty set, then

$$\mathcal{D}_{\mathcal{F}}^* = \text{cone}\left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T, x \in \mathcal{F} \right\}.$$

- If $\text{int}(\mathcal{F}) \neq \emptyset$, then $\mathcal{D}_{\mathcal{F}}^*$ and $\mathcal{D}_{\mathcal{F}}$ are proper.

Properties

- The complexity of checking whether $V \in \mathcal{D}^*_{\mathcal{F}}$ or $U \in \mathcal{D}_{\mathcal{F}}$ depends on $\mathcal{F}$.
- When $\mathcal{F} = \mathbb{R}^n$, $\mathcal{D}^*_{\mathcal{F}} = \mathcal{S}^{n+1}_+$.
- When $\mathcal{F} = \mathbb{R}^n_+$, $\mathcal{D}^*_{\mathcal{F}}$ is the copositive cone!
  Ref: recent survey papers (I. M. Bomze, EJOR, 2012 216(3); Mirjam Dür, Recent Advances in Optimization and its Applications in Engineering, 2010; J.-B. Hiriart-Urruty and A. Seeger, SIAM Review 52(4), 2010.)
- Relaxation or restriction

$$\mathcal{D}^*_{\mathcal{F}} \subseteq \mathcal{S}^n_+ \subseteq \mathcal{D}_{\mathcal{F}}.$$

- Approximation: Computable cover of $\mathcal{F}$.

Checking $U \in \mathcal{D}_{\mathcal{F}}$ is an optimization problem!

$$\mathcal{D}_{\mathcal{F}} = \left\{ U \in \mathcal{S}^{n+1} \middle| \begin{pmatrix} 1 \\ x \end{pmatrix}^T U \begin{pmatrix} 1 \\ x \end{pmatrix} \geq 0, \forall\, x \in \mathcal{F} \right\}.$$

Theorem

$U \in \mathcal{D}_{\mathcal{F}}$ if and only if the optimal value of the following problem is not negative

$$\begin{aligned} \min \quad & \begin{pmatrix} 1 \\ x \end{pmatrix}^T U \begin{pmatrix} 1 \\ x \end{pmatrix} \\ \text{s.t.} \quad & x \in \mathcal{F}. \end{aligned}$$

- If $\mathcal{F}$ is a p-norm constraint, then it is a p-norm constrained quadratic programming.

Easy cases

$$\begin{array}{ll} \min & \begin{pmatrix} 1 \\ x \end{pmatrix}^T U \begin{pmatrix} 1 \\ x \end{pmatrix} \\ \text{s.t.} & x \in \mathcal{F}. \end{array}$$

- If $\mathcal{F}$ is a p-norm constraint, then it is a p-norm constrained quadratic programming.
- When $\mathcal{F} = \{x \in \mathbb{R}^n \mid \frac{1}{2}x^T Px + p^T x + d \leq 0\}$, $P \succ 0$, $int(\mathcal{F}) \neq \emptyset$, it is computable.
- When $\mathcal{F} = Soc(n) = \left\{x \in \mathbb{R}^n | \sqrt{x^T Px} \leq c^T x\right\}$, $P \succ 0$, $int(\mathcal{F}) \neq \emptyset$, it is computable.

A special case of p-norm constrained quadratic programming

$$\begin{aligned} \min \quad & \tfrac{1}{2}x^TQx + q^Tx \\ \text{s.t.} \quad & \|x\|_p \leq k \\ & x \in \mathbb{R}^n, \end{aligned}$$

where $p \geq 1$.

- Equivalent formulation

$$\begin{aligned} \min \quad & \tfrac{1}{2}x^TQx + \tfrac{1}{k}tq^Tx \\ \text{s.t.} \quad & \|x\|_p \leq t \\ & t = k \\ & x \in \mathbb{R}^n. \end{aligned}$$

Homogenous quadratic constrained model

$$\begin{aligned} \min \quad & \tfrac{1}{2}x^TQx + \tfrac{1}{k}tq^Tx \\ \text{s.t.} \quad & \|x\|_p \le t \\ & t = k \\ & x \in \mathbb{R}^n. \end{aligned}$$

- Homogenous quadratic form

$$\begin{aligned} \min \quad & \tfrac{1}{2}\begin{pmatrix} t \\ x \end{pmatrix}^T \begin{pmatrix} 0 & \tfrac{1}{k}q^T \\ \tfrac{1}{k}q & Q \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \\ \text{s.t.} \quad & t = k \\ & \begin{pmatrix} t \\ x \end{pmatrix} \in \left\{(t, x) \in \mathbb{R} \times \mathbb{R}^n \mid \|x\|_p \le t\right\} \end{aligned}$$

Complexity of the problem

- Homogenous: It is polynomially computable when $p = 2$.

$$\begin{array}{ll} \min & x^T Qx \\ \text{s.t.} & x \in Soc(n) = \left\{ x \in \mathbb{R}^n | \sqrt{x^T Px} \leq c^T x \right\}, \end{array}$$

  where Q is a general symmetric matrix, P is positive definite and $Soc(n+1)$ has an interior (Ref: Ye Tian et. al., JIMO 9(3), 2013).
- Variant

$$\begin{array}{ll} \min & x^T Qx + q^T x \\ \text{s.t.} & \|Ax - b\|_2 \leq c^T x \\ & c^T x = d \geq 0 \\ & x \in \mathbb{R}^n. \end{array}$$

- Complexity?

Complexity of the problem

- Homogeneous QP over the 1st-order cone is NP-hard

$$
\begin{array}{ll}
\min & \begin{pmatrix} x_0 \\ x \end{pmatrix}^T Q \begin{pmatrix} x_0 \\ x \end{pmatrix} \\
\text{s.t.} & \begin{pmatrix} x_0 \\ x \end{pmatrix} \in Foc(n+1),
\end{array}
$$

  where $Foc(n+1) = \{(x_0, x) \in \mathbb{R} \times \mathbb{R}^n \mid \|x\|_1 \leq x_0\}$, and Q is a general symmetric matrix.

- It is NP-hard.

Complexity of the problem

- A cross section problem

$$\begin{aligned} \min \quad & \begin{pmatrix} 1 \\ x \end{pmatrix}^T Q \begin{pmatrix} 1 \\ x \end{pmatrix} \\ \text{s.t.} \quad & \|x\|_1 \le 1 \\ & x \in \mathbb{R}^n \end{aligned}$$

- Guo et. al. conjectured NP-hard (Ref: Xiaoling Guo et. al., JIMO 10(3), 2014.
- It is NP-hard (Ref: Yong Hsia, Optimization Letters 8, 2014).

Complexity of the problem

- A general case $p \geq 1$.

$$\begin{aligned} \min \quad & \begin{pmatrix} t \\ x \end{pmatrix}^T Q \begin{pmatrix} t \\ x \end{pmatrix} \\ \text{s.t.} \quad & \begin{pmatrix} t \\ x \end{pmatrix} \in \left\{ (t, x) \in \mathbb{R} \times \mathbb{R}^n \mid \|x\|_p \leq t \right\}. \end{aligned}$$

- Zhou et. al. conjectured NP-hard (Ref: Jing Zhou et. al., PJO to appear, 2014.
- Provided with many solvable subcases.

Quadratic-Function Conic Programming

- PRIMAL

$$\begin{array}{lll} \min & \frac{1}{2}\begin{pmatrix} 2c_0 & q_0^T \\ q_0 & Q_0 \end{pmatrix} \bullet V & \\ \text{s.t.} & \frac{1}{2}\begin{pmatrix} 2c_i & q_i^T \\ q_i & Q_i \end{pmatrix} \bullet V \le 0, i = 1, 2, \ldots, m & (QFCP) \\ & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bullet V = 1 & \\ & V \in \mathcal{D}_{\mathcal{F}}^*. & \end{array}$$

- $\mathcal{F} \subseteq \mathbb{R}^n, A \bullet B = trace(AB^T),$

$$\mathcal{D}_{\mathcal{F}}^* = \mathrm{cl}\left(\mathrm{cone}\left\{\begin{pmatrix} 1 \\ x \end{pmatrix}\begin{pmatrix} 1 \\ x \end{pmatrix}^T, x \in \mathcal{F}\right\}\right).$$

Quadratically Constrained Quadratic Programming (QCQP)

Theorem

If $\mathcal{F} \neq \emptyset$, then the QFCP primal, its dual and the QCQP have the same optimal objective value.

Theorem

Suppose $\mathcal{F}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ be nonempty sets. Denote $v(\mathcal{F})$, $v(\mathcal{G}_1)$ and $v(\mathcal{G}_2)$ be the optimal objective value of the QFCP with $\mathcal{F}$ selecting different sets respectively.
(i) If $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\mathcal{D}_{\mathcal{G}_1} \supseteq \mathcal{D}_{\mathcal{G}_2}$ and $\mathcal{D}^*_{\mathcal{G}_1} \subseteq \mathcal{D}^*_{\mathcal{G}_2}$.
(ii) If $\mathcal{F} \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_2$, then $v(\mathcal{F}) \geq v(\mathcal{G}_1) \geq v(\mathcal{G}_2)$.

Relaxation

- Relaxation
  $\mathcal{C}^* \supseteq \mathcal{D}^*_{\mathcal{F}}$ and computable.

$$
\begin{aligned}
\min \quad & \frac{1}{2}\begin{pmatrix} 2c_0 & q_0^T \\ q_0 & Q_0 \end{pmatrix} \bullet V \\
\text{s.t.} \quad & v_{11} = 1 \\
& \frac{1}{2} H_i \bullet V \leq 0, i = 1, 2, \ldots, s \\
& V = (v_{ij}) \in \mathcal{C}^*,
\end{aligned}
$$

- The worst one: $\mathcal{C}^* = \mathcal{S}^{n+1}_+$.

Ellipsoid Cover of Bounded Feasible Set

- Easy case: Quadratic-function cone over one ellipsoid constraint.

Theorem

Let $\mathcal{F} = \{x \in \mathbb{R}^n \mid g(x) \leqslant 0\}$, *where* $g(x) = \frac{1}{2}x^T Qx + q^T x + c$, $\mathrm{int}(\mathcal{F}) \neq \emptyset$ *and* $Q \in \mathcal{S}^n_{++}$. *For an* $(n+1) \times (n+1)$ *real symmetric matrix* V, $V \in \mathcal{D}^*_{\mathcal{F}}$ *if and only if*

$$\begin{cases} \frac{1}{2}\begin{pmatrix} 2c & q^T \\ q & Q \end{pmatrix} \bullet V \leqslant 0 \\ V \in \mathcal{S}^{n+1}_{+}. \end{cases}$$

Ellipsoid Cover of Bounded Feasible Set

- Ellipsoid cover (Lu et al, 2011)

Theorem

Let $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_s$, *where* $\mathcal{G}_i = \{x \in \mathbb{R}^n \mid \frac{1}{2}x^T B_i x + b_i^T x + d_i \leq 0\}, 1 \leq i \leq s$, *are ellipsoids with an interior, then*

$$\mathcal{D}^*_{\mathcal{G}} = \mathcal{D}^*_{\mathcal{G}_1} + \mathcal{D}^*_{\mathcal{G}_2} + \cdots + \mathcal{D}^*_{\mathcal{G}_s}.$$

And $V \in \mathcal{D}^*_{\mathcal{G}}$ *if and only if the following system is feasible*

$$\begin{cases} V = V_1 + V_2 + \cdots + V_s \\ \frac{1}{2}\begin{pmatrix} 2d_i & b_i^T \\ b_i & B_i \end{pmatrix} \bullet V_i \leqslant 0, i = 1, 2, \ldots, s \\ V_i \in \mathcal{S}^{n+1}_+, i = 1, 2, \ldots, s. \end{cases}$$

Ellipsoid Cover of Bounded Feasible Set

$$
\begin{aligned}
\min\ & H_0 \bullet V \\
s.t.\ & V_{11} = 1 \\
& H_i \bullet V \leqslant 0, i = 1, 2 \ldots, m \\
& V = V_1 + \cdots + V_s \\
& \begin{bmatrix} d_i & b_i^T \\ b_i & B_i \end{bmatrix} \bullet V_i \leqslant 0,\ V_i \succeq 0,\ \ i = 1, 2, ..., s.
\end{aligned} \tag{EC}
$$

It is a SDP, computable!

Ellipsoid Cover: Decomposition

Theorem

Under some assumptions, if $V^* = V_1^* + ... + V_s^*$ *is an optimal solution of (EC), then for each* j, $j = 1, .., s$, *there exists a decomposition of*

$$V_j^* = \sum_{i=1}^{n_j} \mu_{ji} \begin{bmatrix} 1 \\ x_{ji} \end{bmatrix} \begin{bmatrix} 1 \\ x_{ji} \end{bmatrix}^T$$

for some $n_j > 0$, $x_{ji} \in \mathcal{G}_j$, $\mu_{ji} > 0$ *and* $\sum_{i=1}^{n_j} \mu_{ji} = [Y_j^*]_{11}$. *Moreover,* V^* *can be decomposed in the form of*

$$V^* = \sum_{j=1}^{s} \sum_{i=1}^{n_j} \mu_{ji} \begin{bmatrix} 1 \\ x_{ji} \end{bmatrix} \begin{bmatrix} 1 \\ x_{ji} \end{bmatrix}^T$$

with $x_{ji} \in \mathcal{G}_j$, $\mu_{ji} > 0$ *and* $\sum_{j=1}^{s} \sum_{i=1}^{n_j} \mu_{ji} = V_{11}^* = 1$.

Ellipsoid Cover: Approximation Scheme

Step 1 Cover the feasible set $\mathcal{F}$ with some ellipsoid(s).

Step 2 Solve (EC).

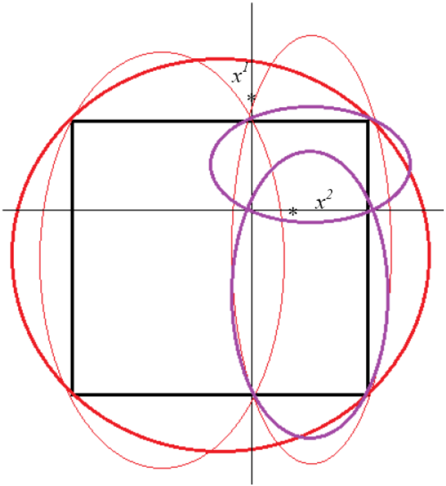
Step 3 Decompose the optimal solution of (EC) and find a x_{ji} with the smallest objective value (sensitive point).

Step 4 Check if the sensitive point $x_{ji} \in \mathcal{F}$. If it is, then it is a global optimum of QCQP. Otherwise, cover $\mathcal{G}_j$ with two smaller ellipsoids. Repeat above procedure.

Step 5 The approximation objective values converge to the optimal value of QCQP.

- Applications: QP (Lu et al, to appear in OPT, 2014), 0-1 knapsack (Zhou et al, JIMO 9(3), 2013), to detect copositve cone (Deng et al, EJOR 229, 2013) etc.

Adaptive ellipsoid covering



Applications to p-norm problems: bounded feasible sets

- p-norm problem

$$\begin{aligned} \min \quad & x^T Qx + q^T x \\ \text{s.t.} \quad & \|x\|_p \leq k \\ & x \in \mathbb{R}^n. \end{aligned}$$

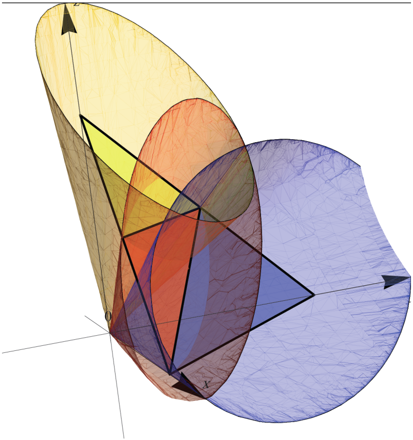
$$\mathcal{F} = \mathcal{D} = \left\{ x \in \mathbb{R}^n \mid \|x\|_p \leq k \right\}.$$

- 2-norm problem

$$\begin{aligned} \min \quad & x^T Qx + q^T x \\ \text{s.t.} \quad & \|Ax - b\|_2 \leq c^T x \\ & c^T x = d, \;\; x \in \mathbb{R}^n. \end{aligned}$$

$$\mathcal{D} = \left\{ x \in \mathbb{R}^n \mid \|Ax - b\|_2 \leq c^T x \right\}, \mathcal{F} = \left\{ x \in \mathcal{D} \mid c^T x = d \right\}.$$

Second-order Cone Cover



Questions

- For the least square problem, why the 2nd-order conic model is not used generally?
- Can we have more efficient algorithms than the interior point method for SDP?

Thank You!