Monotonically Positive Matrices

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Outline of the talk

- Introduction of monotonically positive matrices
- Characterization via $\mathcal{A}$-truncated K-moment problem
- A semidefinite algorithm for checking monotonic positivity
- Examples of monotonically positive matrices

What are monotonically positive matrices?

A real symmetric $n \times n$ matrix A is called monotonically positive (MP) if there exist monotonically nondecreasing vectors $u_1, \cdots, u_m \in \mathbb{R}^n$ such that

$$A = u_1 u_1^T + \cdots + u_m u_m^T.$$

- m is called the length of the decomposition.
- The smallest m is called the MP-rank of A.
- The decomposition is called an MP-decomposition of A.

Denote

$$\mathbb{MR}^n = \{x \in \mathbb{R}^n : x_1 \leq \cdots \leq x_n\}.$$

- A is MP $\Longleftrightarrow$ $A = UU^T$ with $U_i \in \mathbb{MR}^n$.
- A is MP $\Longrightarrow$ $A \in \mathcal{S}_n^+$.

Properties of the MP cone

Denote the MP cone

$$\mathcal{MP}_n := \left\{ \sum_i u_i u_i^T : u_i \in \mathbb{MR}^n \right\}.$$

- It is a proper cone,

  i.e. closed, convex, pointed and full-dimensional.

The dual cone of the MP cone is

$$\mathcal{MP}_n^* := \{B \in \mathcal{S}_n : u^T B u \geq 0 \text{ for all } u \in \mathbb{MR}^n\}.$$

- It is also a proper cone.

Applications in order statistics

- Suppose $(X_1, \ldots, X_n)$ are n jointly distributed random variables. The corresponding order statistics are the $X_i' s$ arranged

$$X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}.$$

- Suppose $x \in \mathbb{R}^n$ is random with $\mathbb{E}x = b$ and the covariance matrix C, where

$$C_{ij} = \mathbb{E}[(x_i - b_i)(x_j - b_j)], \quad i, j = 1, \ldots, n.$$

Let $A \in \mathbb{R}^{n \times n}$ with

$$A_{ij} = \mathbb{E}(x_i x_j) = C_{ij} + b_i b_j, \quad i, j = 1, \ldots, n.$$

Arnold-Balakrishnan-Nagaraja 1992

A basic problem in order statistics

Let $x \in \mathbb{R}^n$ be random, $Ex = b, A_{ij} = E(x_i x_j)$.

Problem:

- Study the probability function of x that is supported in $\mathbb{MR}^n$,
- Whether there exists a finite atomic Borel measure μ supported in $\mathbb{MR}^n$ such that

$$A = \int_{\mathbb{MR}^n} xx^T d\mu = \sum_{i=1}^{m} u_i u_i^T,$$

where $u_i \in \mathbb{MR}^n$ are support points.

- Whether $A \in \mathcal{MP}_n$?

Examples

Example 1. Consider A given as

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

Obviously, A is of rank 1 and

$$A = uu^T = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}^T.$$

Since $u \notin \mathbb{MR}^n$, $A \notin \mathcal{MP}_n$.

Example 2. Consider A given as

$$A = \begin{pmatrix} 5 & 3 & 1 & -5 \\ 3 & 2 & 1 & -3 \\ 1 & 1 & 1 & -1 \\ -5 & -3 & -1 & 5 \end{pmatrix}.$$

Since A can be decomposed as

$$A = u_1 u_1^T + u_2 u_2^T = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \\ 0 \\ 2 \end{pmatrix}^T + \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}^T,$$

where $u_1, u_2 \in \mathbb{MR}^n$, $A \in \mathcal{MP}_n$.

Questions about MP

Given $A \in \mathcal{S}_n$, how to check whether $A \in \mathcal{MP}_n$ or $A \notin \mathcal{MP}_n$?

- If $A \notin \mathcal{MP}_n$, can we get a certificate for this?
- If $A \in \mathcal{MP}_n$, how can we get an MP-decomposition for it?

Identifying vector

A symmetric matrix $A \in \mathcal{S}_n$ can be identified by a vector consisting of its upper triangular entries:

$$\mathrm{vech}(A) = (A_{11}, \ldots, A_{1n}, A_{22}, \ldots, A_{2n}, A_{33}, \ldots, A_{nn})^T.$$

- Let $E := \{(i,j) : 1 \le i \le j \le n\}$. Then, A can also be identified as a vector
$$\mathbf{a} \in \mathbb{R}^E,$$
where $\mathbb{R}^E$ denotes the space of vectors indexed by (i,j) in E.
- For example, $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$,
$E = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$,
$\mathbf{a} = (1,2,1,4,2,1)^T$.

$\mathcal{A}$-truncated moment sequence

Let $\mathbb{N}$ be the set of nonnegative integers.
For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, denote $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Let

$$\mathcal{A} := \{\alpha \in \mathbb{N}^n : |\alpha| = 2\}.$$

- There is a one-to-one correspondence between E and $\mathcal{A}$:

$$(i, j) \;\leftrightarrow\; e_i + e_j.$$

  For example, when $n = 3$,

$$(1, 1) \leftrightarrow (2, 0, 0)^T, \quad (1, 2) \leftrightarrow (1, 1, 0)^T.$$

- $A \in \mathcal{S}_n$ can also be identified as

$$\mathbf{a} = (\mathbf{a}_\alpha)_{\alpha \in \mathcal{A}} \in \mathbb{R}^{\mathcal{A}}, \quad \mathbf{a}_\alpha = A_{ij} \text{ if } \alpha = e_i + e_j \ (i \leq j).$$

  - $\mathbb{R}^{\mathcal{A}}$ denotes the space of vectors indexed by $\alpha \in \mathcal{A}$.
  - $\mathbf{a}$ is called an $\mathcal{A}$-truncated moment sequence ($\mathcal{A}$-tms).

Equivalent condition of MP

Recall that

$$A \in \mathcal{MP}_n \Longleftrightarrow A = u_1 u_1^T + \cdots + u_m u_m^T,\ \ u_i \in \mathbb{MR}^n.$$

Let

$$K = \{x \in \mathbb{R}^n :\ x^T x - 1 = 0, x_1 \le x_2 \le \cdots \le x_n\}.$$

- K is nonempty and compact.
- Every monotonic vector is a multiple of a vector in K.
- $A \in \mathcal{MP}_n \Longleftrightarrow \exists\ \rho_i > 0, u_i \in K$ such that

$$A = \rho_1 u_1 u_1^T + \cdots + \rho_m u_m u_m^T.$$

$\mathcal{A}$-truncated K-moment problem

Let

$$\begin{aligned} \mathcal{A} =& \{\alpha \in \mathbb{N}^n : |\alpha| = 2\}, \\ K =& \{x \in \mathbb{R}^n : x^T x - 1 = 0, x_1 \leq x_2 \leq \cdots \leq x_n\}. \end{aligned}$$

The $\mathcal{A}$-truncated K-moment problem ($\mathcal{A}$-TKMP) is to decide whether $\mathbf{a} \in \mathbb{R}^{\mathcal{A}}$ admits a measure μ on K such that

$$\mathbf{a}_\alpha = \int_K x^\alpha d\mu, \quad \forall \alpha \in \mathcal{A},$$

where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

- μ satisfying the above is called a K-representing measure for $\mathbf{a}$.
- μ is called finitely atomic if its support is a finite set.
- μ is called m-atomic if its support has at most m distinct points.

Lasserre 2001; Nie 2003

Characterization via $\mathcal{A}$-TKMP

Let $\mathcal{A} = \{\alpha \in \mathbb{N}^n : |\alpha| = 2\}$, $K = \{x \in \mathbb{R}^n : x^T x = 1, x_1 \le \cdots \le x_n\}$.
Since

$$\mathbf{a} \in \mathbb{R}^{\mathcal{A}} \text{ admits a } K\text{-measure} \Longleftrightarrow \mathbf{a}_\alpha = \int_K x^\alpha d\mu, \ \forall\, \alpha \in \mathcal{A},$$

Then,

$$A \in \mathcal{MP}_n \Longleftrightarrow A = \rho_1 u_1 u_1^T + \cdots + \rho_m u_m u_m^T.$$

$$\Longleftrightarrow \mathbf{a} \text{ admits an } m\text{-atomic } K\text{-measure, i.e.,}$$

$$\mathbf{a} = \rho_1 [u_1]_{\mathcal{A}} + \cdots + \rho_m [u_m]_{\mathcal{A}},$$

where $\rho_i > 0$, $u_i \in K$, and $[u_i]_{\mathcal{A}} := (u_i^\alpha)_{\alpha \in \mathcal{A}}$.

Denote

$$\mathcal{R}_{\mathcal{A}}(K) = \{\mathbf{a} : \mathbf{a} \text{ admits a } K\text{-measure}\}.$$

Then, $\mathcal{R}_{\mathcal{A}}(K)$ is the MP cone.

K-fullness and $\mathcal{A}$-Riesz function

Let $\mathcal{A} = \{\alpha \in \mathbb{N}^n : |\alpha| = 2\}$, $K = \{x \in \mathbb{R}^n : x^T x = 1, x_1 \le \cdots \le x_n\}$. Denote

$$\mathbb{R}[x]_{\mathcal{A}} := \text{span}\{x^\alpha : \alpha \in \mathcal{A}\} = \text{span}\{x_1^2, x_1 x_2, \cdots, x_n^2\}.$$

- $\mathbb{R}[x]_{\mathcal{A}}$ is called K-full if $\exists p \in \mathbb{R}[x]_{\mathcal{A}}$ such that $p|_K > 0$.
- Choose $p = \sum_{i=1}^n x_i^2 \in \mathbb{R}[x]_{\mathcal{A}}$,
  $p(x) > 0,\ \forall x \in K \implies \mathbb{R}[x]_{\mathcal{A}}$ is K-full.

For $\mathbf{a} \in \mathbb{R}^{\mathcal{A}}$, define an $\mathcal{A}$-Riesz function $\mathcal{L}_{\mathbf{a}}$ acting on $\mathbb{R}[x]_{\mathcal{A}}$ as

$$\mathcal{L}_{\mathbf{a}}(p) := \sum_{\alpha \in \mathcal{A}} p_\alpha a_\alpha, \quad \text{for all } p = \sum_{\alpha \in \mathcal{A}} p_\alpha x^\alpha.$$

Denote $\langle p, \mathbf{a} \rangle := \mathcal{L}_{\mathbf{a}}(p)$ for convenience.

Localizing matrices and moment matrices

Let

$$\mathbb{N}_d^n := \{\alpha \in \mathbb{N}^n : |\alpha| \le d\}, \quad \mathbb{R}[x]_d := \text{span}\{x^\alpha : \alpha \in \mathbb{N}_d^n\}.$$

For $s \in \mathbb{R}^{\mathbb{N}_{2k}^n}$ and $q \in \mathbb{R}[x]_{2k}$, the k-th localizing matrix of q generated by s is the symmetric matrix $L_q^{(k)}(s)$ satisfying

$$\mathcal{L}_s(qp^2) = \text{vec}(p)^T (L_q^{(k)}(s)) \, \text{vec}(p), \; \forall p \in \mathbb{R}[x]_{k-\lceil deg(q)/2 \rceil},$$

- $\text{vec}(p)$ is the coefficient vector of p in the graded lexicographical ordering,
- $\lceil t \rceil$ is the smallest integer that is not smaller than t.
- When $q = 1$, $L_1^{(k)}(s)$ is called a k-th order moment matrix and denoted as $M_k(s)$.

Fialkow-Nie 2012; Helton-Nie 2012; Nie 2013

Localizing matrices and moment matrices

We have

$$L_q^{(k)}(s) = (\sum_\alpha q_\alpha s_{\alpha+\beta+\gamma})_{\beta,\gamma\in\mathbb{N}^n_{k-\lceil deg(q)/2\rceil}},$$

$$M_k(s) = L_1^{(k)}(s) = (s_{\beta+\gamma})_{\beta,\gamma\in\mathbb{N}^n_k}.$$

Denote

$$h(x) := x^T x - 1,$$

$$g(x) := (g_0(x), g_1(x), \ldots, g_{n-1}(x)),$$

where $g_0(x) = 1, g_1(x) = x_2 - x_1, \ldots, g_{n-1}(x) = x_n - x_{n-1}$.
Then, K can be described equivalently as

$$K = \{x \in \mathbb{R}^n : \; h(x) = 0, g(x) \geq 0\}.$$

Example. If $n = 2$ and $k = 2$, the k-th localizing matrix of h generated by s is

$$L^{(2)}_{x_1^2+x_2^2-1}(s) = \begin{pmatrix} s_{(2,0)}+s_{(0,2)}-s_{(0,0)} & s_{(3,0)}+s_{(1,2)}-s_{(1,0)} & s_{(2,1)}+s_{(0,3)}-s_{(0,1)} \\ s_{(3,0)}+s_{(1,2)}-s_{(1,0)} & s_{(4,0)}+s_{(2,2)}-s_{(2,0)} & s_{(3,1)}+s_{(1,3)}-s_{(1,1)} \\ s_{(2,1)}+s_{(0,3)}-s_{(0,1)} & s_{(3,1)}+s_{(1,3)}-s_{(1,1)} & s_{(2,2)}+s_{(0,4)}-s_{(0,2)} \end{pmatrix}.$$

The k-th localizing matrices of $g = (g_0, g_1)$ generated by s are:

$$L^{(2)}_1(s) = M_2(s) = \begin{pmatrix} s_{(0,0)} & s_{(1,0)} & s_{(0,1)} & s_{(2,0)} & s_{(1,1)} & s_{(0,2)} \\ s_{(1,0)} & s_{(2,0)} & s_{(1,1)} & s_{(3,0)} & s_{(2,1)} & s_{(1,2)} \\ s_{(0,1)} & s_{(1,1)} & s_{(0,2)} & s_{(2,1)} & s_{(1,2)} & s_{(0,3)} \\ s_{(2,0)} & s_{(3,0)} & s_{(2,1)} & s_{(4,0)} & s_{(3,1)} & s_{(2,2)} \\ s_{(1,1)} & s_{(2,1)} & s_{(1,2)} & s_{(3,1)} & s_{(2,2)} & s_{(1,3)} \\ s_{(0,2)} & s_{(1,2)} & s_{(0,3)} & s_{(2,2)} & s_{(1,3)} & s_{(0,4)} \end{pmatrix},$$

$$L^{(2)}_{x_2-x_1}(s) = \begin{pmatrix} s_{(0,1)}-s_{(1,0)} & s_{(1,1)}-s_{(2,0)} & s_{(0,2)}-s_{(1,1)} \\ s_{(1,1)}-s_{(2,0)} & s_{(2,1)}-s_{(3,0)} & s_{(1,2)}-s_{(2,1)} \\ s_{(0,2)}-s_{(1,1)} & s_{(1,2)}-s_{(2,1)} & s_{(0,3)}-s_{(1,2)} \end{pmatrix}.$$

Flatness in TKMP

Let $K = \{x \in \mathbb{R}^n : \ h(x) = 0, g(x) \geq 0\}$.

Let $s \in \mathbb{R}^{\mathbb{N}^n_{2k}}$. A necessary condition for s to admits K-measure is

$$L_h^{(k)}(s) = 0, \quad L_{g_j}^{(k)}(s) \succeq 0, \quad j = 0, 1, \ldots, n-1.$$

If, in addition, $\operatorname{rank} M_{k-1}(s) = \operatorname{rank} M_k(s)$, we say s is flat.

Curto-Fialkow (2005) showed

$$s \text{ is flat} \Longrightarrow s \text{ admits a unique } K\text{-measure } \mu$$

i.e.,

$$s = \rho_1 [u_1]_{2k} + \ldots + \rho_m [u_m]_{2k},$$

where $\rho_i > 0$, $u_i \in K$, $m = \operatorname{rank} M_k(s)$, $[u_i]_{2k} := (u_i^\alpha)_{\alpha \in N^n_{2k}}$.

Flat extensions and truncations

Let $\mathcal{A} = \{\alpha \in \mathbb{N}^n : |\alpha| = 2\}, K = \{x \in \mathbb{R}^n : x^T x = 1, x_1 \le \cdots \le x_n\}$.

For $z \in \mathbb{R}^{\mathbb{N}^n_{2k}}$, denote $z|_{\mathcal{A}} = (z_\alpha)_{\alpha \in \mathcal{A}}$.

- If $\mathbf{a} = z|_{\mathcal{A}}$, call z is an extension of $\mathbf{a}$, or $\mathbf{a}$ is a truncation of z.
- If $\mathbf{a} = z|_{\mathcal{A}}$ and z is flat, call z is a flat extension of $\mathbf{a}$.
- If $\mathbf{a} = z|_{\mathcal{A}}$ and $\mathbf{a}$ is flat, call $\mathbf{a}$ is a flat truncation of z.

Fact: If $\mathbf{a} \in \mathbb{R}^{\mathcal{A}}$ has a flat extension, then $\mathbf{a}$ admits a K-measure:

$$\mathbf{a} = z|_{\mathcal{A}}, z \text{ is flat} \Longrightarrow z = \int_K [x]_{2k} d\mu \Longrightarrow \mathbf{a} = \int_K [x]_{\mathcal{A}} d\mu.$$

Flat extensions and measures

The following statements are equivalent:

$$\begin{aligned} A \in \mathcal{MP}_n &\iff \mathbf{a} \text{ admits a } K\text{-measure} \\ &\iff \mathbf{a} \text{ admits a } m\text{-atomic } K\text{-measure, with } m \leq |\mathcal{A}| \\ &\iff \mathbf{a} \text{ has a flat extension.} \end{aligned}$$

To check MP, it is enough to find a flat extension.

Questions: How to decide if $\mathbf{a}$ has a flat extension?

- If yes, how to find it?
- If no, how can we get a certificate?

Nie 2012.

Linear optimization with moments

Let $d > 2$ be an even integer. Choose $F(x) \in \mathbb{R}[x]_d$,

$$F(x) = \sum_{\alpha \in \mathbb{N}_d^n} F_\alpha x^\alpha.$$

Consider the linear optimization problem with moments:

$$(P): \quad \begin{aligned} \eta = \min_{z} \quad & \langle F, z \rangle \\ \text{s.t.} \quad & z|_{\mathcal{A}} = \mathbf{a}, z \in \mathcal{R}_d(K), \end{aligned}$$

where

$$\mathcal{R}_d(K) = \{z \in \mathbb{R}^{\mathbb{N}_d^n} : \ z \text{ admits a } K\text{-measures}\}.$$

Choices of $F(x)$

Note that

$$(P): \qquad \begin{aligned} \eta = \min_{z} \quad & \langle F, z \rangle \\ \text{s.t.} \quad & z|_{\mathcal{A}} = \mathbf{a}, z \in \mathcal{R}_d(K), \end{aligned}$$

- Since K is compact, $\mathbb{R}[x]_{\mathcal{A}}$ is K-full,
  $\Longrightarrow \mathcal{F}(P)$ is compact convex,
  $\Longrightarrow (P)$ has a minimizer for all F.
- Choose $F \in \Sigma_{n,d}$, the set of all sum of squares polynomials in n variables with degree d.

Semidefinite relaxations

Recall $z \in \mathbb{R}^{\mathbb{N}^n_{2k}}$ is flat, if

$$L_h^{(k)}(z) = 0, \quad L_{g_j}^{(k)}(z) \succeq 0, \quad j = 0, 1, \ldots, n-1,$$
$$\operatorname{rank} M_{k-1}(z) = \operatorname{rank} M_k(z).$$

Denote

$$\Gamma_{\mathcal{A}}^k(K) = \left\{ z \in \mathbb{R}^{\mathbb{N}^n_{2k}} : L_h^{(k)}(z) = 0, L_{g_j}^{(k)}(z) \succeq 0, j = 0, 1, \cdots, n \right\},$$
$$\Upsilon_{\mathcal{A}}^k(K) = \left\{ z|_{\mathcal{A}} : z \in \Gamma_{\mathcal{A}}^k(K) \right\}.$$

If $k < deg(\mathcal{A})/2$, $\Upsilon_{\mathcal{A}}^k(K)$ is defined to be $\mathbb{R}^{\mathcal{A}}$, by default. Then,

$$\Upsilon_{\mathcal{A}}^1(K) \supseteq \cdots \supseteq \Upsilon_{\mathcal{A}}^k(K) \supseteq \Upsilon_{\mathcal{A}}^{k+1}(K) \supseteq \cdots \supseteq \mathcal{R}_{\mathcal{A}}(K)$$

and

$$\bigcap_{k=1}^{\infty} \Upsilon_{\mathcal{A}}^k(K) = \mathcal{R}_{\mathcal{A}}(K).$$

Semidefinite relaxations

Let $\Gamma_{\mathcal{A}}^{k}(K) = \left\{ z \in \mathbb{R}^{\mathbb{N}_{2k}^{n}} : L_{h}^{(k)}(z) = 0, L_{g_j}^{(k)}(z) \succeq 0, j = 0, 1, \cdots, n \right\}$.

$$(P): \quad \begin{array}{rl} \eta = \min\limits_{z} & \langle F, z \rangle \\ \text{s.t.} & z|_{\mathcal{A}} = \mathbf{a}, z \in \mathcal{R}_d(K). \end{array}$$

The k-th order semidefinite relaxation of (P) is

$$(SDR)_k: \quad \begin{array}{rl} \eta_k = \min\limits_{z} & \langle F, z \rangle \\ \text{s.t.} & z|_{\mathcal{A}} = \mathbf{a}, z \in \Gamma_{\mathcal{A}}^{k}(K). \end{array}$$

Suppose $z^{*,k}$ is a minimizer of $(SDR)_k$.

- $\eta^k \leq \eta$ for all k.
- If $\mathbf{a} = z^{*,k}|_{\mathcal{A}} \in \mathcal{R}_{\mathcal{A}}(K)$, then $\eta^k = \eta$, $z^{*,k}$ is minimizer of (P), i.e., the relaxation $(SDR)_k$ is exact for solving (P).

A certificate for $\mathbf{a}$ admitting no measure

Let $\mathbf{a} \in \mathbb{R}^{\mathcal{A}}$, $K = \{x : h(x) = 0, (g_0(x), \ldots, g_{n-1}(x)) \geq 0\}$,

Fact: Since $\mathcal{F}(P) \subseteq \mathcal{F}((SDR)_k)$, $\mathbf{a}$ admits no K-measure if the semidefinite relaxation

$$\begin{aligned} L_h^{(k)}(z) = 0, L_{g_j}^{(k)}(z) \succeq 0, j = 0, 1, \ldots, n-1 \\ z|_{\mathcal{A}} = \mathbf{a}, z \in \mathbb{R}^{\mathbb{N}_{2k}^n} \end{aligned}$$

is infeasible for any $k \geq \deg(\mathcal{A})/2$.

Remark. Suppose K is compact and $\mathbb{R}[x]_{\mathcal{A}}$ is K-full. If $\mathbf{a}$ admits no K-measure, then for some k the above semidefinite relaxation is infeasible.

Nie, 2002

A semidefinite algorithm for checking MP

Step 0 Choose $F \in \Sigma_{n,d}$, let $k := 2$.

Step 1 Solve the k-th relaxation problem $(SDR)_k$.
If it is infeasible, $\mathbf{a}$ doesn't admit a K-measure.
Otherwise, compute a minimizer $z^{*,k}$. Let $t := 1$.

Step 2 Let $w := z^{*,k}|_{2t}$.
If the rank condition is not satisfied, go to Step 4.

Step 3 Compute the finitely atomic measure μ admitted by w:

$$\mu = \rho_1 \delta(u_1) + \cdots + \rho_m \delta(u_m),$$

where $\rho_i > 0$, $u_i \in K$, $m = \operatorname{rank} M_t(w)$, $\delta(u_i)$ is the Dirac measure supported on the point u_i. Stop.

Step 4 If $t < k$, set $t := t + 1$, go to Step 2;
otherwise, set $k := k + 1$, go to Step 1.

Properties of the algorithm

The algorithm has the following properties:

- If $(SDR)_k$ is infeasible for some k, $A \notin \mathcal{MP}_n$.
- If $A \notin \mathcal{MP}_n$, the $(SDR)_k$ is infeasible for all k big enough.
- If $A \in \mathcal{MP}_n$, for almost all generated F, we can asymptotically get an MP-decomposition of A, by solving the hierarchy of $(SDR)_k$ for $k = 2, 3, \ldots$.

Remark (finite convergence).

- If $A \in \mathcal{MP}_n$, under some general conditions, which is almost necessary and sufficient, an MP-decomposition of A can often be obtained within finitely many steps.
- This always happens in numerical experiments.

Numerical experiments

- Choose $F = [x]_{d/2}^T H^T H [x]_{d/2}$, where $[x]_{d/2} := (x^\alpha)_{\alpha \in \mathbb{N}^n_{d/2}}$, H is a random square matrix obeying Gaussian distribution.
- $(SDR)_k$ is solved by GloptiPoly 3 and SeDuMi.
- The rank condition is checked numerically with SVD.

  The rank of a matrix is evaluated as the number of its singular values that are greater than or equal to 10^{-6}.
- Henrion & Lasserre's method is used to compute an m-atomic K-measure for w.

Example 1. Consider A given as

$$A = \begin{pmatrix} 8 & 6 & 6 & 2 & 2 & 0 & 0 \\ 6 & 5 & 4 & 1 & 0 & -1 & -2 \\ 6 & 4 & 5 & 2 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 2 & 0 & 3 & 2 & 5 & 3 & 6 \\ 0 & -1 & 1 & 1 & 3 & 2 & 4 \\ 0 & -2 & 2 & 2 & 6 & 4 & 8 \end{pmatrix}.$$

- A has rank 2.

- A can be decomposed as

$$A = u_1 u_1^T + u_2 u_2^T = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}^T + \begin{pmatrix} -2 \\ -2 \\ -1 \\ 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} -2 \\ -2 \\ -1 \\ 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}^T,$$

  where $u_1 \notin \mathbb{MR}^n$, $u_2 \in \mathbb{MR}^n$.

- The algorithm terminates at Step 1 with $k = 2$,
  i.e. $(SDR)_k$ is infeasible.

  So, $A \notin \mathcal{MP}_n$.

Example 2. Consider A given as

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

- A is symmetric diagonally dominant, $A \in \mathcal{S}_n^+$.
- The algorithm terminates at Step 1 with $k = 2$,
  i.e. $(SDR)_k$ is infeasible.

  So, $A \notin \mathcal{MP}_n$.

Example 3. Consider A given as

$$A = \begin{bmatrix} 6 & 3 & 0 & -3 & -6 \\ 3 & 2 & 1 & 0 & -1 \\ 0 & 1 & 2 & 3 & 4 \\ -3 & 0 & 3 & 6 & 9 \\ -6 & -1 & 4 & 9 & 14 \end{bmatrix}.$$

Since

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}^T + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \end{bmatrix}^T + \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}^T,$$

$A \in \mathcal{MP}_n$.

- The algorithm terminates at Step 3 with $k = 3$.
- It gives an MP-decomposition $A = \sum_{i=1}^{2} \rho_i u_i u_i^T$ with

$$\rho_1 = 7.5000, \quad u_1 = (-0.4472, -0.4472, -0.4472, -0.4472, -0.4472)^T,$$
$$\rho_2 = 22.5000, \quad u_2 = (-0.4472, -0.1491, 0.1491, 0.4472, 0.7454)^T.$$

- The length of the MP-decomposition is shorter, which shows an advantage of the algorithm.

Example 4. Consider A given as

$$A = \begin{bmatrix} 290 & 221 & 195 & 102 & -100 & -188 & -289 \\ 221 & 170 & 152 & 76 & -86 & -150 & -222 \\ 195 & 152 & 139 & 64 & -91 & -143 & -198 \\ 102 & 76 & 64 & 40 & -20 & -52 & -100 \\ -100 & -86 & -91 & -20 & 108 & 118 & 110 \\ -188 & -150 & -143 & -52 & 118 & 172 & 194 \\ -289 & -222 & -198 & -100 & 110 & 194 & 290 \end{bmatrix}.$$

Since A has the decomposition

$$A = \begin{bmatrix} -9 \\ -6 \\ -4 \\ -4 \\ -3 \\ 3 \\ 8 \end{bmatrix} \begin{bmatrix} -9 \\ -6 \\ -4 \\ -4 \\ -3 \\ 3 \\ 8 \end{bmatrix}^T + \begin{bmatrix} -9 \\ -7 \\ -7 \\ -2 \\ 7 \\ 9 \\ 9 \end{bmatrix} \begin{bmatrix} -9 \\ -7 \\ -7 \\ -2 \\ 7 \\ 9 \\ 9 \end{bmatrix}^T + \begin{bmatrix} -8 \\ -7 \\ -7 \\ -2 \\ 7 \\ 9 \\ 9 \end{bmatrix} \begin{bmatrix} -8 \\ -7 \\ -7 \\ -2 \\ 7 \\ 9 \\ 9 \end{bmatrix}^T + \begin{bmatrix} -8 \\ -6 \\ -5 \\ -4 \\ 1 \\ 1 \\ 8 \end{bmatrix} \begin{bmatrix} -8 \\ -6 \\ -5 \\ -4 \\ 1 \\ 1 \\ 8 \end{bmatrix}^T,$$

$A \in \mathcal{MP}_n$.

- The algorithm terminates at Step 3 with $k = 3$.
- It gives an MP-decomposition $A = \sum_{i=1}^{4} \rho_i u_i u_i^T$ with

$$\rho_1 = 127.9059,\ u_1 = (-0.5860, -0.3835, -0.2417, -0.2417, -0.2417, 0.2732, 0.5096)^T,$$
$$\rho_2 = 226.0146,\ u_2 = (-0.5817, -0.4083, -0.3053, -0.3053, -0.0864, 0.0120, 0.5485)^T,$$
$$\rho_3 = 478.9378,\ u_3 = (-0.4361, -0.3729, -0.3655, -0.1258, 0.3350, 0.4208, 0.4818)^T,$$
$$\rho_4 = 376.1435,\ u_4 = (-0.4569, -0.3532, -0.3516, -0.1017, 0.3463, 0.4542, 0.4542)^T.$$

Conclusions

- We introduce the MP matrices.
- We formulate the problem of checking MP as a linear optimization with moments.
- A semidefinite algorithm is proposed for checking whether a given symmetric matrix A is MP or not.
  - If $A \notin \mathcal{MP}_n$, we can give a certificate.
  - If $A \in \mathcal{MP}_n$, we can give an MP-decomposition for it.

Thank you very much!