Estimating Dynamic Discrete-Choice Games of Incomplete Information

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joint work with Michael Egesdal and Zhenyu Lai (Harvard University)

2014 Workshop on Optimization for Modern Computation
BICMR
September 2–4, 2014
Roadmap of the Talk

- Introduction / Literature Review
- The Model
- Estimation
- Monte Carlo Experiments / Results
- Conclusion
Part I

Introduction
Discrete-Choice Games

• An active research topic in applied econometrics, empirical IO and marketing
• Classical application: entry/exit decisions
  • Bresnahan and Reiss (1987, 1991), Berry (1992)
  • Determining the sources of firms profitability
  • Understanding how firms react to competition
• Other applications:
  • Location choices: Seim (2006), Orhun (2012)
  • Technology innovation: Igami (2012)
• Identification: Sweeting (2009), de Paula and Tang (2012)
Extry/Exit Games: An Illustrating Example

- Five firms: $i = 1, \ldots, 5$
- Firm $i$’s decision in period $t$:
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Estimation Methods for Discrete-Choice Games of Incomplete Information

- Maximum-Likelihood (ML) estimator
  - Efficient estimator in large-sample theory
  - Expensive to compute
  - Computationally simple
  - Potentially large finite-sample biases
- Moment inequality estimator: Pakes, Porter, Ho, and Ishii (2011)
  - does not require the assumption that only one equilibrium is played in the data
- Constrained optimization approach: Su and Judd (2012), Dubé, Fox and Su (2012)
What We Do in This Paper

- Based on Su and Judd (2012), propose a constrained optimization formulation for the ML estimator to estimate dynamic games
- Conduct Monte Carlo experiments to compare performance of different estimators
  - Two-step pseudo maximum likelihood (2S-PML) estimator
  - NPL estimator implemented by NPL algorithm and NPL-Λ algorithm
  - ML estimator via the constrained optimization approach
Part II

The Model
The Dynamic Game Model in AM (2007)

- **Discrete time infinite-horizon**: $t = 1, 2, \ldots, \infty$
- **$N$ players**: $i \in \mathcal{I} = \{1, \ldots, N\}$
- **The market is characterized by size** $s^t \in \mathcal{S} = \{s_1, \ldots, s_L\}$.
  - market size is observed by all players
  - exogenous and stationary market size transition: $f_S(s^{t+1}|s^t)$
- **At the beginning of each period $t$**, player $i$ observes $(x^t, \varepsilon_i^t)$
  - $x^t$: a vector of common-knowledge state variables
  - $\varepsilon_i^t$: private shocks
- **Players then simultaneously choose whether to be active in the market in that period**
  - $a^t_i \in \mathcal{A} = \{0, 1\}$: player $i$’s action in period $t$
  - $a^t = (a_1^t, \ldots, a_N^t)$: the collection of all players’ actions.
  - $a_{-i}^t = (a_1^t, \ldots, a_{i-1}^t, a_{i+1}^t, \ldots, a_N^t)$: the current actions of all players other than $i$
State Variables

- **Common-knowledge state variables:** $x^t = (s^t, a^{t-1})$
- **Private shocks:** $\varepsilon_i^t = \{\varepsilon_i^t(a_i^t)\}_{a_i^t \in A}$
  - $\varepsilon_i^t(a_i^t)$ has a i.i.d type-I extreme value distribution across actions and players as well as over time.
  - opposing players know only its probability density function $g(\varepsilon_i^t)$.
- **The conditional independence** assumption on state transition:
  $$p[x^{t+1} = (s', a'), \varepsilon_i^{t+1} | x^t = (s, \tilde{a}), \varepsilon_i^t, a^t] = f_S(s'|s)1\{a' = a^t\}g(\varepsilon_i^{t+1})$$
Player $i$’s Utility Maximization Problem

- $\theta$: the vector of structural parameters
- $\beta \in (0, 1)$: the discount factor.
- player $i$’s per-period payoff function:
  \[
  \tilde{\Pi}_i (a_i^t, a_{-i}^t, x^t, \varepsilon_i^t; \theta) = \Pi_i (a_i^t, a_{-i}^t, x^t; \theta) + \varepsilon_i^t (a_i^t)
  \]
- The common-knowledge component of the per-period payoff
  \[
  \Pi_i (a_i^t, a_{-i}^t, x^t; \theta) = \begin{cases} 
  \theta^{RS} s_t - \theta^{RN} \log \left( 1 + \sum_{j \neq i} a_j^t \right) - \theta^{FC} i - \theta^{EC} \left( 1 - a_i^{t-1} \right), & \text{if } a_i^t = 1, \\
  0, & \text{if } a_i^t = 0,
  \end{cases}
  \]
- Player $i$’s utility maximization problem:
  \[
  \max_{\{a_i^t, a_i^{t+1}, a_i^{t+2}, \ldots\}} \mathbb{E} \left[ \sum_{\tau=t}^{\infty} \beta^{\tau-t} \tilde{\Pi}_i (a_i^\tau, a_{-i}^\tau, x^\tau, \varepsilon_i^\tau; \theta) \mid (x^t, \varepsilon_i^t) \right]
  \]
Equilibrium Concept: Markov Perfect Equilibrium

- Equilibrium characterization in terms of the observed states $\mathbf{x}$
- $P_i(a_i|\mathbf{x})$: the conditional choice probability of player $i$ choosing action $a_i$ at state $\mathbf{x}$
- $V_i(\mathbf{x})$: the expected value function for player $i$ at state $\mathbf{x}$
- Define $P = \{P_i(a_i|\mathbf{x})\}_{i \in \mathcal{I}, a_i \in \mathcal{A}, \mathbf{x} \in \mathcal{X}}$ and $V = \{V_i(\mathbf{x})\}_{i \in \mathcal{I}, \mathbf{x} \in \mathcal{X}}$
- A Markov perfect equilibrium is a vector $(V, P)$ that satisfies two systems of nonlinear equations:
  - Bellman equation (for each player $i$)
  - Bayes-Nash equilibrium conditions
System I: Bellman Optimality

- **Bellman Optimality.** \( \forall i \in I, x \in X \)

  \[
  V_i(x) = \sum_{a_i \in A} P_i(a_i|x) \left[ \pi_i(a_i|x, \theta) + e_i^P(a_i, x) \right] + \beta \sum_{x' \in X} V_i(x') f^P_{X}(x'|x)
  \]

- \( \pi_i(a_i|x, \theta) \): the expected payoff of \( \Pi_i(a_i, a_{-i}, x; \theta) \) for player \( i \) from choosing action \( a_i \) at state \( x \) and given \( P_j(a_j|x) \),

  \[
  \pi_i(a_i|x, \theta) = \sum_{a_{-i} \in A^{N-1}} \left\{ \prod_{a_j \in a_{-i}} P_j(a_j|x) \right\} \Pi_i(a_i, a_{-i}, x; \theta)
  \]

- \( f^P_{X}(x'|x) \): state transition probability of \( x \), given \( P \)

  \[
  f^P_{X}[x' = (s', a')|x = (s, \tilde{a})] = \prod_{j=1}^{N} P_j(a'_j|x) \quad f_S(s'|s)
  \]

- \( e_i^P(a_i, x) = \text{Euler’s Constant} - \sigma \log \left[ P_i(a_i|x) \right] \)
System II: Bayes-Nash Equilibrium Conditions

- **Bayes-Nash Equilibrium.**

\[
P_i (a_i = j | \mathbf{x}) = \frac{\exp [v_i (a_i = j | \mathbf{x})]}{\sum_{k \in \mathcal{A}} \exp [v_i (a_i = k | \mathbf{x})]}, \quad \forall i \in \mathcal{I}, j \in \mathcal{A}, \mathbf{x} \in \mathcal{X},
\]

- **\( v_i (a_i | \mathbf{x}) \):** choice-specific expected value function

\[
v_i (a_i | \mathbf{x}) = \pi_i (a_i | \mathbf{x}, \theta) + \beta \sum_{\mathbf{x}' \in \mathcal{X}} V_i (\mathbf{x}') f_i^P (\mathbf{x}' | \mathbf{x}, a_i)
\]

- **\( f_i^P (\mathbf{x}' | \mathbf{x}, a_i) \):** the state transition probability conditional on the current state \( \mathbf{x} \), player \( i \)'s action \( a_i \), and his beliefs \( P \)

\[
f_i^P [\mathbf{x}' = (s', a') | \mathbf{x} = (s, \tilde{a}), a_i] = f_S (s' | s) 1 \{a'_i = a_i\} \prod_{j \in \mathcal{I} \setminus i} P_j (a'_j | \mathbf{x})
\]
Markov Perfect Equilibrium

- **Bellman Optimality.** \( \forall i \in I, x \in X \)

\[
V_i(x) = \sum_{a_i \in A} P_i(a_i|x) \left[ \pi_i(a_i|x, \theta) + e^P_i(a_i, x) \right] + \beta \sum_{x' \in X} V_i(x') f^P(x'|x)
\]

- **Bayes-Nash Equilibrium.**

\[
P_i(a_i = j|x) = \frac{\exp[v_i(a_i = j|x)]}{\sum_{k \in A} \exp[v_i(a_i = k|x)]}, \quad \forall i \in I, j \in A, x \in X,
\]

- In compact notation

\[
V = \Psi^V(V, P, \theta)
\]

\[
P = \Psi^P(V, P, \theta)
\]

- Set of all Markov Perfect Equilibria

\[
SOL(\Psi, \theta) = \left\{ (P, V) \mid V = \Psi^V(V, P, \theta), P = \Psi^P(V, P, \theta) \right\}
\]
Part III

Estimation
Data Generating Process

- $\theta^0$: the true value of structural parameters in the population
- $(V^0, P^0)$: a Markov perfect equilibrium at $\theta^0$
- **Assumption**: If multiple Markov perfect equilibria exist, only one equilibrium is played in the data
- **Data**: $Z = \{\bar{a}^{mt}, \bar{x}^{mt}\}_{m \in \mathcal{M}, t \in T}$
  - observations from $M$ independent markets over $T$ periods
  - In each market $m$ and time period $t$, researchers observe
    - the common-knowledge state variables $\bar{x}^{mt}$
    - players’ actions $\bar{a}^{mt} = (\bar{a}_1^{mt}, \ldots, \bar{a}_N^{mt})$
Maximum-Likelihood Estimation

- For a given $\theta$, let $(P^\ell(\theta), V^\ell(\theta)) \in SOL(\Psi, \theta)$ be the $\ell$-th equilibrium.
- Given data $Z = \{\bar{a}^{mt}, \bar{x}^{mt}\}_{m \in M, t \in T}$, the logarithm of the likelihood function is

$$L(Z, \theta) = \max_{(P^\ell(\theta), V^\ell(\theta)) \in SOL(\Psi, \theta)} \frac{1}{M} \sum_{i=1}^{N} \sum_{m=1}^{M} \sum_{t=1}^{T} \log P^\ell_i (\bar{a}^{mt}_i | \bar{x}^{mt})(\theta)$$

- The ML estimator is

$$\theta^{ML} = \arg\max_{\theta} L(Z, \theta)$$  \hspace{1cm} (1)
NFXP’s Likelihood as a Function of $(\alpha, \beta) - \text{Eq 1}$
NFXP’s Likelihood as a Function of \((\alpha, \beta)\) – Eq 2
NFXP’s Likelihood as a Function of $(\alpha, \beta)$ – Eq 3
ML Estimation via Constrained Optimization Approach

• Given data $Z = \{\bar{a}^{mt}, \bar{x}^{mt}\}_{m \in M, t \in T}$, the logarithm of the augmented likelihood function is

$$\mathcal{L}(Z, P) = \frac{1}{M} \sum_{i=1}^{N} \sum_{m=1}^{M} \sum_{t=1}^{T} \log P_i(\bar{a}^{mt}_i|\bar{x}^{mt}) .$$

• The constrained optimization formulation of the ML estimation problem is

$$\max_{(\theta, P, V)} \mathcal{L}(Z, P) \quad \text{subject to} \quad V = \Psi^V (V, P, \theta) \quad P = \Psi^P (V, P, \theta) \quad (2)$$

• **Proposition 1.** Problem (1) and (2) have the same solution.
Asymptotic Properties of ML Estimator via Constrained Optimization Approach

- **Theorem.** The constrained maximum likelihood estimator is consistent and asymptotic normal.

See Appendix.

Aitchison and Silvey (1958) and Section 10.3 in Gourieroux and Monfort (1995).
Two-Step Methods: Intuition

- Recall the constrained optimization formulation for the ML estimator is

\[
\max_{(\theta, P, V)} \mathcal{L}(Z, P)
\]
subject to
\[
V = \Psi^V(V, P, \theta)
\]
\[
P = \Psi^P(V, P, \theta)
\]

- Denote the solution by \((\theta^*, P^*, V^*)\)

- Suppose we know \(P^*\), how do we recover \(\theta^*\) (and \(V^*\))?
Two-Step Pseudo Maximum-Likelihood (2S-PML)

• Step 1: nonparametrically estimate the conditional choice probabilities, denoted by $\hat{P}$ directly from the observed data $Z$.

• Step 2: Solve

$$\max_{(\theta, P, V)} \mathcal{L}(Z, P)$$

subject to

$$V = \Psi^V \left( V, \hat{P}, \theta \right)$$

$$P = \Psi^P \left( V, \hat{P}, \theta \right)$$

or, equivalently,

$$\max_{(\theta, V)} \mathcal{L} \left( Z, \Psi^P \left( V, \hat{P}, \theta \right) \right)$$

subject to

$$V = \Psi^V \left( V, \hat{P}, \theta \right)$$
Reformulation of the Optimization Problem in Step 2

- **Bellman Optimality.** \( \forall i \in \mathcal{I}, x \in \mathcal{X} \)

\[
V_i(x) = \sum_{a_i \in A} P_i(a_i \mid x) \left[ \pi_i(a_i \mid x, \theta) + e_i^P(a_i, x) \right] + \beta \sum_{x' \in \mathcal{X}} V_i(x') f_{\mathcal{X}}(x' \mid x)
\]
Reformulation of the Optimization Problem in Step 2

- **Bellman Optimality.** \( \forall i \in \mathcal{I}, x \in \mathcal{X} \)

\[
V_i(x) = \sum_{a_i \in \mathcal{A}} P_i(a_i|x) \left[ \pi_i(a_i|x, \theta) + e_i^P(a_i, x) \right] + \beta \sum_{x' \in \mathcal{X}} V_i(x') f_{\mathcal{X}}^P(x'|x)
\]

- Define \( V_i = [V_i(x)]_{x \in \mathcal{X}}, \hat{P}_i(a_i) = [\hat{P}_i(a_i|x)]_x, e_i^\hat{P}(a_i) = [e_i^\hat{P}(a_i, x)]_x, \pi_i(a_i, \theta) = [\pi_i(a_i|x, \theta)]_x, \) and \( F_{\mathcal{X}}^{\hat{P}} = \left[ f_{\mathcal{X}}^{\hat{P}}(x'|x) \right]_{x, x' \in \mathcal{X}} \)
Reformulation of the Optimization Problem in Step 2

- **Bellman Optimality.** \( \forall i \in I, x \in X \)

\[
V_i(x) = \sum_{a_i \in A} P_i(a_i|x) \left[ \pi_i(a_i|x, \theta) + e_i^P(a_i, x) \right] + \beta \sum_{x' \in X} V_i(x') f_X^P(x'|x)
\]

- Define \( V_i = [V_i(x)]_{x \in X} \), \( \hat{P}_i(a_i) = [\hat{P}_i(a_i|x)]_{x} \), \( e_i^P(a_i) = [e_i^P(a_i, x)]_{x} \), \( \pi_i(a_i, \theta) = [\pi_i(a_i|x, \theta)]_{x} \), and \( F_X^\hat{P} = [f_X^\hat{P}(x'|x)]_{x, x' \in X} \)

- The Bellman equation above can be rewritten as

\[
\left[I - \beta F_X^{\hat{P}}\right] V_i = \sum_{a_i \in A} \left[ \hat{P}_i(a_i) \circ \pi_i(a_i, \theta) \right] + \sum_{a_i \in A} \left[ \hat{P}_i(a_i) \circ e_i^P(a_i) \right],
\]

or equivalently

\[
V_i = \left[I - \beta F_X^{\hat{P}}\right]^{-1} \left\{ \sum_{a_i \in A} \left[ \hat{P}_i(a_i) \circ \pi_i(a_i, \theta) \right] + \sum_{a_i \in A} \left[ \hat{P}_i(a_i) \circ e_i^P(a_i) \right] \right\},
\]

or in a compact notation

\[
V = \Gamma(\theta, \hat{P}).
\]
Reformulation of the Optimization Problem in Step 2

- Replacing the constraint $V = \Psi^V (V, \hat{P}, \theta)$ by $V = \Gamma(\theta, \hat{P})$ through a simple elimination of variables $V$, the optimization problem in Step 2 becomes

$$\max_{\theta} \mathcal{L} \left( Z, \Psi^P \left( \Gamma(\theta, \hat{P}), \hat{P}, \theta \right) \right).$$

- The 2S-PML estimator is defined as

$$\theta^{2S-PML} = \arg\max_{\theta} \mathcal{L} \left( Z, \Psi^P \left( \Gamma(\theta, \hat{P}), \hat{P}, \theta \right) \right).$$
The 2S-PML estimator can have large biases in finite samples. In an effort to reduce the finite-sample biases associated with the 2S-PML estimator, Aguirregabiria and Mira (2007) propose an NPL estimator. An NPL fixed point \((\tilde{\theta}, \tilde{P})\) satisfies the conditions:

\[
\tilde{\theta} = \arg\max_{\theta} \mathcal{L} \left( Z, \Psi^P \left( \Gamma(\theta, \tilde{P}), \tilde{P}, \theta \right) \right)
\]

\[
\tilde{P} = \Psi^P \left( \Gamma(\tilde{\theta}, \tilde{P}), \tilde{P}, \tilde{\theta} \right)
\]

(3)
**NPL Algorithm**

- The NPL algorithm: For $1 \leq K \leq \bar{K}$, iterate over Steps 1 and 2 below until convergence:

  **Step 1.** Given $\tilde{P}_{K-1}$, solve
  \[
  \tilde{\theta}_K = \arg\max_{\theta} \mathcal{L} \left( Z, \Psi^{P} \left( \Gamma(\theta, \tilde{P}_{K-1}), \tilde{P}_{K-1}, \theta \right) \right).
  \]

  **Step 2.** Given $\tilde{\theta}_K$, update $\tilde{P}_K$ by
  \[
  \tilde{P}_K = \Psi^{P} \left( \Gamma(\tilde{\theta}_K, \tilde{P}_{K-1}), \tilde{P}_{K-1}, \tilde{\theta}_K \right); \text{ increase } K \text{ by } 1
  \]

- Convergence criterion:
  \[
  \left\| (\tilde{\theta}_K, \tilde{P}_K) - (\tilde{\theta}_{K-1}, \tilde{P}_{K-1}) \right\| \leq \text{tol}_NPL
  \]
  \[
  \text{tol}_NPL: \text{ the convergence tolerance, for example, } 1.0\text{e}^{-6}
  \]

- If the NPL algorithm converges, $(\tilde{\theta}_K, \tilde{P}_{K-1})$ approximately satisfies the NPL fixed-point conditions (3):
  \[
  \left\| \tilde{P}_{K-1} - \Psi^{P} \left( \Gamma(\tilde{\theta}_K, \tilde{P}_{K-1}), \tilde{P}_{K-1}, \tilde{\theta}_K \right) \right\| \leq \text{tol}_NPL
  \]
A Modified NPL Algorithm: NPL-Λ

- It is now well known that the NPL algorithm may not converge or even if it converges, it may fail to provide consistent estimates; Pesendorfer and Schmidt-Dengler (2010)
- Kasahara and Shimotsu (2012) propose the NPL-Λ algorithm that modifies Step 2 of the NPL algorithm to compute the NPL estimator

$$\tilde{P}_K = \left( \Psi^P \left( \Gamma(\tilde{\theta}_K, \tilde{P}_{K-1}), \tilde{P}_{K-1}, \tilde{\theta}_K \right) \right)^{\lambda} \left( \tilde{P}_{K-1} \right)^{1-\lambda}$$

where $\lambda$ is chosen to be between 0 and 1
  - $\lambda = 0$: two-step PML estimator
  - $\lambda = 1$: NPL algorithm
- The proper value for $\lambda$ depends on the true parameter values $\theta^0$
- Alternatively, Kasahara and Shimotsu suggest using a small number for the spectral radius
Convergence Criteria for the NPL-Λ Algorithm

• The NPL-Λ algorithm: For $1 \leq K \leq \bar{K}$, iterate over Steps 1 and 2 below until convergence:

**Step 1.** Given $\tilde{P}_{K-1}$, solve

$$\tilde{\theta}_K = \arg\max_{\theta} \mathcal{L} \left( Z, \Psi^P \left( \Gamma(\theta, \tilde{P}_{K-1}), \tilde{P}_{K-1}, \theta \right) \right).$$

**Step 2.** Given $\tilde{\theta}_K$, update $\tilde{P}_K$ by

$$\tilde{P}_K = \left( \Psi^P \left( \Gamma(\tilde{\theta}_K, \tilde{P}_{K-1}), \tilde{P}_{K-1}, \tilde{\theta}_K \right) \right)^\lambda \left( \tilde{P}_{K-1} \right)^{1-\lambda};$$

increase $K$ by 1

• Convergence criterion used in Kasahara and Shimotsu (2012):

$$\left\| (\tilde{\theta}_K, \tilde{P}_K) - (\tilde{\theta}_{K-1}, \tilde{P}_{K-1}) \right\| \leq \text{tol}_{\text{NPL}}$$

• If the NPL-Λ algorithm converges, does $(\tilde{\theta}_K, \tilde{P}_{K-1})$ approximately satisfy the NPL fixed-point conditions (3)?

$$\left\| \tilde{P}_{K-1} - \Psi^P \left( \Gamma(\tilde{\theta}_K, \tilde{P}_{K-1}), \tilde{P}_{K-1}, \tilde{\theta}_K \right) \right\| \leq \text{tol}_{\text{NPL}}??$$
Convergence Criteria for the NPL-Λ Algorithm

- Using the previous convergence criterion, if the NPL-Λ algorithm converges,

\[
\| \tilde{P}_{K-1} - \Psi^P \left( \Gamma(\tilde{\theta}_K, \tilde{P}_{K-1}), \tilde{P}_{K-1}, \tilde{\theta}_K \right) \| \leq \frac{\text{tol}_{NPL}}{\lambda}
\]

- If one uses a very small value for \( \lambda \), e.g., \( \lambda = 1.0 \times 10^{-5} \), and \( \text{tol}_{NPL} = 1.0 \times 10^{-6} \), then \( \frac{\text{tol}_{NPL}}{\lambda} = 0.1 \)

- Appropriate convergence criterion:

\[
\| \left( \tilde{\theta}_K, \tilde{P}_K \right) - \left( \tilde{\theta}_{K-1}, \tilde{P}_{K-1} \right) \| \leq \text{tol}_{NPL}.
\]

\[
\| \tilde{P}_{K-1} - \Psi^P \left( \Gamma(\tilde{\theta}_K, \tilde{P}_{K-1}), \tilde{P}_{K-1}, \tilde{\theta}_K \right) \| \leq \text{tol}_{NPL}.
\]
Part IV

Monte Carlo
Monte Carlo
Experiment Design

- Three experiment specifications with two cases in each experiment
- Experiment 1: Kasahara and Shimotsu (2012) example
- Experiment 2: Aguirregabiria and Mira (2007) example
- Experiment 3: Examples with increasing \(|S|\), the number of market size values

- Market size transition matrix is

$$f_S(s^{t+1} | s^t) = \begin{pmatrix}
0.8 & 0.2 & 0 & \ldots & 0 & 0 \\
0.2 & 0.6 & 0.2 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0.2 & 0.6 & 0.2 \\
0 & 0 & \ldots & 0 & 0.2 & 0.8
\end{pmatrix}$$
Experiment 2: Aguirregabiria and Mira (2007) Example

- \( N = 5 \) players
- \( S = \{1, 2, \ldots, 5\} \)
- Total number of grid points in the state space:
  \[ |X| = |S| \times |A|^N = 5 \times 2^5 = 160 \]
- The discount factor \( \beta = 0.95 \); the scale parameter of the type-I extreme value distribution \( \sigma = 1 \)
- The common-knowledge component of the per-period payoff

\[
\Pi_i \left( a^t_i, a^t_{-i}, x^t ; \theta \right) = \begin{cases} 
\theta_{RS} s^t - \theta_{RN} \log \left( 1 + \sum_{j \neq i} a^t_j \right) - \theta_{FC,i} - \theta_{EC} \left( 1 - a^t_{i-1} \right), & \text{if } a^t_i = 1, \\
0 & \text{if } a^t_i = 0,
\end{cases}
\]

- \( \theta = (\theta_{RS}, \theta_{RN}, \theta_{FC}, \theta_{EC}) \): the vector of structural parameters with
  \( \theta_{FC} = \{\theta_{FC,i}\}_{i=1}^N \)
Experiment 2: Cases 3 and 4

- True values of structural parameters $\theta_{FC}^0 = (1.9, 1.8, 1.7, 1.6, 1.5)$ and $\theta_{EC}^0 = 1$
- Consider two sets of true parameter values for $\theta_{RS}$ and $\theta_{RN}$
  
  Case 3:  $(\theta_{RN}^0, \theta_{RS}^0) = (2, 1)$;
  
  Case 4:  $(\theta_{RN}^0, \theta_{RS}^0) = (4, 2)$.

- Case 3 is Experiment 3 in Aguirregabiria and Mira (2007)
- The ML estimator solves the constrained optimization problem with 2,400 constraints and 2,408 variables.
Experiment 3: Cases 5 and 6

- Consider two sets of market size values:
  
  Case 5:  $|S| = 10$ with $S = \{1, 2, \ldots, 10\}$;  
  
  Case 6:  $|S| = 15$ with $S = \{1, 2, \ldots, 15\}$.

- All other specifications remain the same as those in Case 3 in Experiment 2
- Case 5: The ML estimator solves the constrained optimization problem with 4,800 constraints and 4,808 variables.
- Case 6: The ML estimator solves the constrained optimization problem with 7,200 constraints and 7,208 variables.
Data Simulation and Algorithm Implementation

- Data simulation: MATLAB
- Optimization: AMPL (programming language) / KNITRO (NLP solver), providing first-order / second-order analytical derivatives
- In each data set: $M = 400$ and $T = 10$
- For Case 3 and 4 in Experiments 2
  - Construct 100 data sets for each case
  - 10 starting points for each data set
- For Cases 5 and 6 in Experiments 3
  - Construct 50 data sets for each case
  - 5 start points for each data sets
- For NPL and NPL-$\Lambda$: $\bar{K} = 100$
- For the NPL-$\Lambda$ algorithm: $\lambda = 0.5$
Monte Carlo Results: Percentage of Data Sets Solved

<table>
<thead>
<tr>
<th>Success Probability</th>
<th>Percentage of Data Sets Solved</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>Case NPL</td>
</tr>
<tr>
<td>25%</td>
<td>Case NPL-Lambda</td>
</tr>
<tr>
<td>50%</td>
<td>MLE</td>
</tr>
<tr>
<td>75%</td>
<td>2S-PML</td>
</tr>
<tr>
<td>100%</td>
<td>NPL</td>
</tr>
</tbody>
</table>

Case

- Case NPL
- Case NPL-Lambda
- MLE
- 2S-PML
- NPL

Success Probability

- 0%
- 25%
- 50%
- 75%
- 100%

Case

- 3
- 4
- 5
- 6
Monte Carlo Results: Avg. Solve Time Per Run

![Graph showing Monte Carlo results with solve time per run on the y-axis and case numbers on the x-axis. The graph includes data points for MLE, 2S-PML, NPL, and NPL-Lambda.]
## Monte Carlo Results: Estimates for Experiment 2

<table>
<thead>
<tr>
<th>Case</th>
<th>Estimator</th>
<th>$\theta_{FC,1}$</th>
<th>$\theta_{FC,2}$</th>
<th>$\theta_{FC,3}$</th>
<th>$\theta_{FC,4}$</th>
<th>$\theta_{FC,5}$</th>
<th>$\theta_{EC}$</th>
<th>$\theta_{RN}$</th>
<th>$\theta_{RS}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>MLE</td>
<td>1.895 (0.077)</td>
<td>1.794 (0.078)</td>
<td>1.697 (0.075)</td>
<td>1.597 (0.074)</td>
<td>1.495 (0.073)</td>
<td>0.990 (0.046)</td>
<td>2.048 (0.345)</td>
<td>1.011 (0.095)</td>
</tr>
<tr>
<td>3</td>
<td>2S-PML</td>
<td>1.884 (0.066)</td>
<td>1.774 (0.069)</td>
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## Monte Carlo Results: Estimates for Experiment 3

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<td>1.012 (0.000)</td>
<td>1.992 (0.000)</td>
<td>1.007 (0.000)</td>
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Implementation Improvements and Robustness Checks

- **ML estimator**
  Can we improve the performance (reduce computational time) of the constrained optimization approach for the ML estimator?
  - Use 2S-PML estimates as starting values for the constrained optimization problem for the ML estimator

- **NPL-Λ algorithm**
  Can we improve the convergence results of the NPL-Λ algorithm by using different values for \( \lambda \) or \( \bar{K} \)?
  - Use \( \lambda \in \{0.1, 0.3, 0.5, 0.7, 0.9\} \)
ML Estimator/Constr. Opt. using 2S-PML Estimates as Starting Values for Cases 3 and 4

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ML Estimator/Constr. Opt. using 2S-PML Estimates as Starting Values for Cases 5 and 6

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### NPL-Λ Algorithm using Different λ Values for Case 4

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## NPL-Λ Algorithm using Different λ Values for Case 6

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</tr>
</tbody>
</table>
Final Comment

- Lyapunov-Stable Equilibria
  - Aguirregabiria and Nevo (2012) have argued that with multiple equilibria, it is reasonable to assume that only Lyapunov-stable (or best-response stable) equilibria will be played in the data, in which case the NPL algorithm should converge
  - Lyapunov-stable (or best-response stable) equilibria:

\[ \rho \left( \nabla_{P} \Psi^{P} \left( \Gamma(\theta^{0}, P^{0}), P^{0}, \theta^{0} \right) \right) < 1 \]

- The spectral radius of the mapping above depends not only on \( \theta^{0} \) but also on the grid of the market size values, market size transition, etc.

- Ongoing work:
  - Robustness check for NPL-\( \Lambda \) algorithm with different choices of \( \lambda \) value
  - Performance of other two-step estimators?
  - Improving the performance of the constrained optimization approach on dynamic games with higher-dimensional state space?