Exact Recovery for Sparse Signal
via Weighted $\ell_1$ Minimization

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Outline

1 Introduction
2 Weighted Null Space Property
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1 Introduction

1.1 Background

The concept of compressed sensing was first introduced by Donoho [D], Candès, Romberg and Tao [CRT] and Candès and Tao [CT]. Since then myriads of researchers have been lured to this area owing to its wide applications in signal processing, communications, astronomy, biology, medicine and so forth, see, e.g., [EK].

1 Introduction
1.2 Problem

The fundamental problem in compressed sensing is to recover a sparse solution $x \in \mathbb{R}^n$ of the underdetermined system of the form

$$\Phi x = y,$$

where $y \in \mathbb{R}^m$ is the available measurement and $\Phi \in \mathbb{R}^{m \times n}$ is a known measurement matrix.
1 Introduction

1.3 Model Representation

To recover a sparse solution $x \in \mathbb{R}^n$ of the form $\Phi x = y$, the underlying model is the following $\ell_0$ minimization:

$$\min \|x\|_0, \quad \text{s.t. } \Phi x = y,$$

(1)

where $\|x\|_0$ is $\ell_0$-norm of the vector $x \in \mathbb{R}^n$. However (1) is NP-Hard.
1 Introduction
1.3 Model Representation

One common approach is to solve (1) via convex $\ell_1$ minimization:

$$\min \| x \|_1, \quad \text{s.t. } \Phi x = y.$$  \hspace{2cm} (2)

The use of $\ell_1$ minimization has become so extensively that it could arguably be considered *the modern least squares*, see, e.g., [BDE],[CWX] and [CZ].


Inspired by the efficiency of $\ell_1$ minimization, it is natural to ask, for example, whether a different (but perhaps again convex) alternative to $\ell_0$ minimization might also find the correct solution, but with a lower measurement requirement than $\ell_1$ minimization.

Numerical experiments indicate that the reweighted $\ell_1$ minimization does outperform unweighted $\ell_1$ minimization in many situations.

In this talk, as a sequence, we consider the theoretical properties of the \textit{weighted ℓ₁ minimization}:

$$\min \| w \odot x \|_1, \; \text{s.t.} \; \Phi x = b,$$

where $\odot$ denotes the Hadamard product, that is $\| w \odot x \|_1 = \sum \omega_i |x_i|$, and $0 < \omega_i \leq 1$, $i = 1, 2, \ldots, n$. 
1 Introduction

1.3 Model Representation

Some cases that $\ell_1$ minimization will fail to recover the sparse signal while exact recovery can be succeeded via weighted $\ell_1$ minimization. (a) Sparse signal $x^{(0)} = (0, 0, 2)^T$, feasible set $\Phi x = b$, and in $\ell_1$ ball there exists an $x^{(1)} = (\frac{3}{4}, \frac{3}{4}, 0)^T$ but $\|x^{(1)}\|_0 > \|x^{(0)}\|_0$. (b) In weighted $\ell_1$ ball, there does not exist an $x \neq x^{(0)}$ such that $\|x\|_0 \leq \|x^{(0)}\|_0$. 
1 Introduction
1.4 Null Space Property

The null space property (NSP) is the necessary and sufficient condition for (2) to reconstruct the system \( b = \Phi x \) exactly, see, e.g., [Z].

**Definition I.1 (NSP)**

A matrix \( \Phi \in \mathbb{R}^{m \times n} \) satisfies the null space property of order \( k \) if for all subsets \( S \in \mathcal{C}_n^k \) it holds

\[
\| h_S \|_1 < \| h_{Sc} \|_1
\]

for any \( h \in \mathcal{N}(\Phi) \setminus \{0\} \), where \( \mathcal{N}(\Phi) = \{ h \in \mathbb{R}^n \mid \Phi h = 0 \} \) and \( \mathcal{C}_n^k = \{ S \subset \{1, 2, \cdots, n\} \mid |S| = k \} \).

Another most popular sufficient condition for exact sparse recovery is *Restricted Isometry Property* (RIP) introduced by Candès and Tao [CT].

**Definition 1.2 (RIP)**

For $k \in \{1, 2, \cdots, n\}$, the restricted isometry constant is the smallest positive number $\delta_k$ such that

$$(1 - \delta_k)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta_k)\|x\|_2^2$$

hold for all $k$-sparse vector $x \in \mathbb{R}^n$, i.e., $\|x\|_0 \leq k$.

1 Introduction

1.7 Current Results for $\ell_1$ Minimization

<table>
<thead>
<tr>
<th></th>
<th>$\delta_k$</th>
<th>$\delta_{2k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Candès</td>
<td>- -</td>
<td>0.4142</td>
</tr>
<tr>
<td>Foucart and Lai</td>
<td>- -</td>
<td>0.4531</td>
</tr>
<tr>
<td>Foucart</td>
<td>- -</td>
<td>0.4652</td>
</tr>
<tr>
<td>Cai, Wang and Xu</td>
<td>- -</td>
<td>0.4721</td>
</tr>
<tr>
<td>Mo and Li</td>
<td>- -</td>
<td>0.4931</td>
</tr>
<tr>
<td>Cai and Zhang</td>
<td>1/3</td>
<td>0.5000</td>
</tr>
<tr>
<td>Zhou, Kong and Xiu</td>
<td>- -</td>
<td>0.5746</td>
</tr>
<tr>
<td></td>
<td></td>
<td>with $\delta_{8k} &lt; 1$</td>
</tr>
<tr>
<td>Andersson and Strömberg</td>
<td>- -</td>
<td>0.6246</td>
</tr>
</tbody>
</table>

Table: Different bounds on $\delta_k$ and $\delta_{2k}$.

Recently, Cai and Zhang [CZ] got a sharp bound

$$\delta_{tk} < \sqrt{\frac{t - 1}{t}}.$$  \hspace{1cm} (6)

☆ Particularly, $\delta_{2k} < \frac{\sqrt{2}}{2}$. It is worth mentioning that (6) is the sharp bound for $\ell_1$ minimization which has been proved in [CZ].

As for the weighted $\ell_1$ minimization, literature [FMSY] presented us the upper bound on $\delta_k$ might be $\delta_k < 0.4343$ under some cases.

Definition II.1

A matrix $\Phi \in \mathbb{R}^{m \times n}$ satisfies the null space property of order $k$ if for all subsets $S \in \mathcal{C}^k_n$ it holds

$$\|h_S\|_1 < \|h_{S^c}\|_1$$

(7)

for any $h \in \mathcal{N}_1 := \{h \in \mathbb{R}^n | h \in \mathcal{N}(\Phi), \|h\|_1 = 1\}$.

Lemma II.2

Definition I.1 is equivalent to Definition II.1.
2 Weighted Null Space Property
2.2 Property of the WNSP

**Definition II.2 (WNSP)**

For a given weight $\omega \in \mathbb{R}^n$, a matrix $\Phi \in \mathbb{R}^{m \times n}$ satisfies the weighted null space property of order $k$ if for all subsets $S \in \mathcal{C}^k_n$ it holds

$$\| \omega \circ h_S \|_1 < \| \omega \circ h_{Sc} \|_1 \quad (8)$$

for any $h \in \mathcal{N}_1$.

**Theorem II.2**

Every $k$-sparse vector $\hat{x} \in \mathbb{R}^n$ is the unique solution of the weighted minimization (3) with $b = \Phi \hat{x}$ iff $\Phi$ satisfies the WNSP of order $k$. 
2 Weighted Null Space Property

2.3 Two Examples

\[ \Phi = \begin{pmatrix} 4/5 & 0 & 3/10 \\ 0 & 4/5 & 3/10 \end{pmatrix}, \quad b = \begin{pmatrix} 3/5 \\ 3/5 \end{pmatrix}. \]

Clearly, the unique solution of \( \ell_0 \) and \( \ell_1 \) models are 
\[ x^{(0)} = (0, 0, 2)^T \text{ and } x^{(1)} = (3/4, 3/4, 0)^T. \]

If setting \( \omega_2 = \omega_1, \omega_3 < \frac{3}{4} \omega_1 \), \( x^{(0)} \) is also the unique solution of the weighted \( \ell_1 \) model.

For any \( h \in \mathcal{N}_1 \), we have 
\[ h = \left( \frac{3}{8} h_3, \frac{3}{8} h_3, -h_3 \right)^T \text{ with } h_3 = 4/7. \]
Then for all subset \( S \in \mathcal{C}_3^1 \) and the given \( \omega \) it holds 
\[ \| \omega \circ h_S \|_1 < \| \omega \circ h_{Sc} \|_1, \] which means \( \Phi \) satisfies WNSP. It is worth mentioning that this \( \Phi \) does not satisfy the NSP due to 
\[ |h_3| \not\approx |\frac{3}{4} h_3| = |h_1| + |h_2|. \]
Some cases that $\ell_1$ minimization will fail to recover the sparse signal while exact recovery can be succeeded via weighted $\ell_1$ minimization. (a) Sparse signal $x^{(0)} = (0, 0, 2)^T$, feasible set $\Phi x = b$, and in $\ell_1$ ball there exists an $x^{(1)} = \left(\frac{3}{4}, \frac{3}{4}, 0\right)^T$ but $\|x^{(1)}\|_0 > \|x^{(0)}\|_0$. (b) In weighted $\ell_1$ ball, there does not exist an $x \neq x^{(0)}$ such that $\|x\|_0 \leq \|x^{(0)}\|_0$. 
2 Weighted Null Space Property

2.3 Two Examples

\[ \Phi = \begin{pmatrix} 3/4 & -1/2 & 3/8 & 1/2 & -1/4 \\ 3/4 & -1/2 & -1/8 & 1/2 & 0 \\ 0 & 1/4 & 3/8 & -1/8 & -3/8 \end{pmatrix}, \quad b = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/8 \end{pmatrix}. \]

\[ x^{(0)} = (0, 0, 0, 1, 0)^T, \quad x^{(1)} = \left( \frac{1}{3}, -\frac{1}{2}, 0, 0, 0 \right)^T, \]
\[ \omega_2 = \frac{2}{3} \omega_1, \omega_4 = \frac{1}{2} \omega_1, \omega_3 = \omega_5 = \omega_1, \]
\[ h = \left( \frac{-8h_2 + 13h_5}{12}, h_2, \frac{h_5}{2}, \frac{4h_2 - 3h_5}{2}, h_5 \right)^T. \]

Likely, \( \Phi \) satisfies the WNSP we defined while does not content the NSP.
3 Restricted Isometry Property

3.1 Design the Weight

We first design a way of weighing and introduce some notations. Let $T_0$ and $\hat{h}$ be the optimal solution of the following model

\[
(T_0, \hat{h}) := \arg\max_{T \in C_n^k, h \in \mathbb{N}_1} \|h_T\|_1.
\]

(9)

For a constant $0 < \gamma \leq 1$, we define $\omega$ based on $T_0$ as

\[
\omega_i = \begin{cases} 
\gamma, & i \in T_0, \\
1, & i \in T_0^c,
\end{cases}
\]

(10)

where $T_0^c$ is the complementary set of $T_0$ in $\{1, 2, \cdots, n\}$.
Lemma III.1

Let $T_0$ and $\hat{h}$ be defined as (9). If $T_0$ uniquely exists, then there exists $\omega$ defined as (10) with $0 < \gamma < 1$ such that

$$\|\omega \circ \hat{h}_{T_0}\|_1 = \max_{T \in \mathcal{C}_n^k, h \in \mathcal{N}_1} \|\omega \circ h_T\|_1.$$  \hspace{1cm} (11)

If $T_0$ exists but not uniquely, then $\omega$ defined as (10) with $\gamma = 1$ that satisfies (11).
### Theorem III.2

For the given $\gamma$ and $\omega$ as (9) and (10), if

$$\delta_{ak} < \sqrt{\frac{a - 1}{a - 1 + \gamma^2}}$$  \hspace{1cm} (12)

holds for some $a > 1$, then each $k$ sparse minimizer $\hat{x}$ of the weighted $\ell_1$ minimization (3) is the solution of (1).
3 Restricted Isometry Property

3.3 Main Theorems

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\delta_{2k}$</th>
<th>$\delta_{3k}$</th>
<th>$\delta_{4k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\sqrt{2}/2$</td>
<td>$\sqrt{6}/3$</td>
<td>$\sqrt{3}/2$</td>
</tr>
<tr>
<td>3/4</td>
<td>0.800</td>
<td>0.883</td>
<td>0.917</td>
</tr>
<tr>
<td>1/2</td>
<td>0.894</td>
<td>0.942</td>
<td>0.960</td>
</tr>
<tr>
<td>1/4</td>
<td>0.970</td>
<td>0.984</td>
<td>0.989</td>
</tr>
</tbody>
</table>

Table: Bounds on $\delta_{2k}$, $\delta_{3k}$ and $\delta_{4k}$ with different cases.
Theorem III.3

For the given \( \gamma \) and \( \omega \) as (9) and (10), if

\[
\delta_k < \begin{cases} 
\frac{1}{1 + 2\lceil \gamma k \rceil / k}, & \text{for even number } k \geq 2, \\
\frac{1}{1 + 2\lceil \gamma k \rceil / \sqrt{k^2 - 1}}, & \text{for odd number } k \geq 3,
\end{cases}
\]

holds, where \( \lceil a \rceil \) denotes the smallest integer that is no less than \( a \), then each \( k \) sparse minimizer \( \hat{x} \) of the weighted \( \ell_1 \) minimization (3) is the solution of (1).
### 3 Restricted Isometry Property

#### 3.3 Main Theorems

Table: Bounds on $\delta_k$ with different cases. From the table one cannot difficultly find that under some mild situation, the upper bounds are greater than 0.4343.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$k \geq 2$ is even</th>
<th>$k \geq 3$ is odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/3</td>
<td>0.3203</td>
</tr>
<tr>
<td>3/4</td>
<td>3/8 ($k \geq 4$)</td>
<td>0.3797 ($k \geq 5$)</td>
</tr>
<tr>
<td>1/2</td>
<td>1/2 ($k \geq 2$)</td>
<td>$\sqrt{6} - 2$ ($k \geq 5$)</td>
</tr>
<tr>
<td>1/4</td>
<td>2/3 ($k \geq 4$)</td>
<td>$3 - \sqrt{6}$ ($k \geq 5$)</td>
</tr>
<tr>
<td>1/6</td>
<td>3/4 ($k \geq 6$)</td>
<td>0.7101 ($k \geq 5$)</td>
</tr>
</tbody>
</table>
3 Restricted Isometry Property
3.4 Two Examples

\[ \Phi = \begin{pmatrix} 4/5 & 0 & 3/10 \\ 0 & 4/5 & 3/10 \end{pmatrix}, \quad b = \begin{pmatrix} 3/5 \\ 3/5 \end{pmatrix}. \]

From \( h = \begin{pmatrix} 3/8 h_3, 3/8 h_3, -h_3 \end{pmatrix}^T \in \mathcal{N}_1 \) with \( h_3 = 4/7 \), \( |h_3| \) is the largest entry of \( h \), i.e. \( T_0 = \{3\} \) uniquely exists. Therefore by setting \( 3/8 < \omega_3 = \gamma < 0.418, \omega_1 = \omega_2 = 1 \), we have \( \gamma \| h_{\{3\}} \|_1 < \| h_{\{1,2\}} \|_1 \), which means that \( x^{(0)} \) is the unique solution of weighted \( \ell_1 \) model. We directly calculate that \( \delta_2 = 0.9224 \) with \( n = 3, k = 2 \) by the following formula

\[ \delta_k = \max_{S \in \mathcal{C}_n^k} \| \Phi^T S \Phi S - I_k \|, \]

(15)

where \( \| \cdot \| \) denotes the spectral norm of a matrix. Since \( T_0 \) uniquely exists and \( \gamma < 0.418 \), it yields \( \delta_2 < 0.9226 \) from (12) by taking \( a = 2, k = 1 \). Hence the \( \ell_0 \) minimization can be exactly reconstructed by the weighted \( \ell_1 \) minimization from our Theorem III.2
3 Restricted Isometry Property

3.4 Two Examples

\[ \Phi = \begin{pmatrix} 3/4 & -1/2 & 3/8 & 1/2 & -1/4 \\ 3/4 & -1/2 & -1/8 & 1/2 & 0 \\ 0 & 1/4 & 3/8 & -1/8 & -3/8 \end{pmatrix}, \quad b = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/8 \end{pmatrix}. \]

From \( h = \left( \frac{-8h_2 + 13h_5}{12}, h_2, \frac{h_5}{2}, \frac{4h_2 - 3h_5}{2}, h_5 \right)^T \), it follows that

\[ T_0 = \{4\}, \quad \hat{h} = (-2h_2/3, h_2, 0, 2h_2, 0)^T, \quad h_2 = 6/11, \]

which manifests that \( T_0 \) uniquely exists. By setting \( \omega_4 = \gamma = 0.3, \omega_1 = \omega_2 = \omega_3 = \omega_5 = 1 \), we have \( \gamma \|h_\{4\}\|_1 < \|h_\{1,2,3,5\}\|_1 \), which means that \( x^{(0)} \) is the unique solution of weighted \( \ell_1 \) minimization. We compute \( \delta_2 = 0.9572 \) by (15) with \( n = 5, k = 2 \). Since \( T_0 \) uniquely exists and \( \gamma = 0.3 \), it yields \( \delta_2 < 0.9578 \) from (12) by taking \( a = 2, k = 1 \). And thus the \( \ell_0 \) minimization can be exactly recovered via the weighted \( \ell_1 \) minimization from Theorem III.2.
Although $T_0$ defined by (9) always exists but not uniquely sometimes. However, from Examples above, we can see the assumption that $T_0$ uniquely exists is actually not a strong assumption to a certain extent.
The relationship between WNSP, NSP and RIP, the dashed area denotes the scale of matrices that satisfy the RIP via weighted $\ell_1$ minimization.
Thank you!