Invariant theory:
classical, quantum and super

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Plan

1. The 3 formulations of invariant theory
2. Classical and quantum invariant theory for $GL_n$
3. Classical and quantum invariant theory for $O_n$ and $Sp_n$
4. The problem which Brauer left
5. The second fundamental theorem–solution of Brauer’s problem in the classical cases.
6. From classical to quantum-the $BMW$ algebra
8. Invariant theory for supergroups.
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The three formulations of invariant theory

Let $G$ be a connected reductive group over $\mathbb{C}$, and $V$ a representation of $G$.

First: Describe $\text{End}_G(V^\otimes r)$ for each $r$. [Non-commutative algebra]

Second: Describe $((\otimes^r V^*) \otimes (\otimes^s V))^G$. [Multilinear algebra]

Third: Let $W = \bigoplus^\ell V \bigoplus^m V^*$. Give a presentation of $\mathbb{C}[W]^G$
Geometrically: Describe $W//G$. [Commutative algebra, GIT]

In each context, these are all equivalent, by similar, but various arguments
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In each case the problem divides into:

(i) Find generators (FFT)
(ii) Give all relations among these (SFT)

There are very few pairs \((G, V)\) for which satisfactory answers are known

The modern version of the subject might be said to have started with Gauss’ *Disquisitiones Arithmeticae* in 1802.

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$g.v_1 \otimes \ldots \otimes v_r := gv_1 \otimes \ldots \otimes gv_r \ (g \in \text{GL}(V), v_i \in V)$.

For $\pi \in \text{Sym}_r$, $\mu_r(\pi) : v_1 \otimes \ldots \otimes v_r \mapsto v_{\pi^{-1}(1)} \otimes \ldots \otimes v_{\pi^{-1}(r)}$
lies in $\text{End}_G(V^\otimes r)$

Hence we have $\mu_r : \mathbb{C}\text{Sym}_r \to \text{End}_G(V^\otimes r)$.

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For any \( r \), let \( a(r) = \sum_{\pi \in \text{Sym}_r} \varepsilon(\pi) \pi \) (\( \text{Sym}_r \)-alternator)

Since \( a(n + 1) V^{\otimes n+1} \subseteq \wedge^{n+1}(V) = 0 \), clearly \( a(n + 1) \in \text{Ker}(\mu_r) \).

\[
\text{SFT(Schur) Ker}(\mu_r) = \langle a(n + 1) \rangle. \mu_r \text{ is an isomorphism if } r < n + 1.
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Both FFT and SFT are most easily proved using semisimplicity, but are valid much more generally.

These statements may be easily translated into their equivalents in the multilinear and commutative algebra formulations.
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Quantum theory-type $A$

Let $K = \mathbb{C}(q)$, $q$ an indeterminate, $\mathcal{U}_q = U_q(g\mathfrak{l}_n)$, a $K$-Hopf algebra.

$\mathcal{C}$: the category of f.d. $g\mathfrak{l}_n(\mathbb{C})$-modules.

If $\mathcal{C}_q$ is the category of f.d. $\mathcal{U}_q$-modules of type $(1, 1, ..., 1)$, we have a weight-preserving equivalence $\mathcal{C} \xrightarrow{\sim} \mathcal{C}_q$, where $v \in W_q \subset \mathcal{C}_q$ has weight $\lambda$ if $K_i v = q^{\langle \alpha_i, \lambda \rangle} v$.

There is a universal $R$-matrix $\check{R} \in \mathcal{T}(\mathcal{U}_q \otimes \mathcal{U}_q)$ such that for any $V_q \in \mathcal{C}_q$:

(i) $R := P \check{R} \in \text{End}_{\mathcal{U}_q}(V_q \otimes V_q)$, where $P (v \otimes w) = w \otimes v$.

(ii) If $R_i = R$ acting on the $(i, i + 1)$ factors of $V_q^\otimes r$, then $R_i R_j = R_j R_i$ if $|i - j| \geq 2$, and $R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$.

Hence we have $\mu_{r, q} : KB_r \to \text{End}_{\mathcal{U}_q}(V_q^\otimes r)$. for any $V_q \in \mathcal{C}_q$, where $B_r$ is the $r$-string braid group.
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(ii) If $R_i = R$ acting on the $(i, i + 1)$ factors of $V_q^\otimes r$, then $R_i R_j = R_j R_i$ if $|i - j| \geq 2$, and $R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$.

Hence we have $\mu_{r,q} : K\mathcal{B}_r \to \text{End}_{\mathcal{U}_q}(V_q^\otimes r)$, for any $V_q \in \mathcal{C}_q$, where $\mathcal{B}_r$ is the $r$-string braid group.
Quantum theory-type A

Let $K = \mathbb{C}(q)$, $q$ an indeterminate, $U_q = U_q(\mathfrak{gl}_n)$, a $K$-Hopf algebra.

$C$: the category of f.d. $\mathfrak{gl}_n(\mathbb{C})$-modules.

If $C_q$ is the category of f.d. $U_q$-modules of type $(1,1,...,1)$, we have a weight-preserving equivalence $C \sim C_q$, where $v \in W_q \in C_q$ has weight $\lambda$ if $K_i v = q^{\langle \alpha_i, \lambda \rangle} v$.

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FFT: If $V_q$ is the ‘natural module’ for $U_q(\mathfrak{gl}_n)$, $\mu_{r,q}$ is surjective.

This is easily proved directly from the classical case using integral forms of $U_q$ and $V_q$.

Next, observe that if $\varepsilon_1, \ldots, \varepsilon_n$ are the standard weights, $V_q = L_{\varepsilon_1, q}$, and $V_q \otimes V_q \simeq L_{2\varepsilon_1, q} \oplus L_{\varepsilon_1 + \varepsilon_2, q}$.

Known: $R$ acts on $L_{2\varepsilon_1, q}$ and $L_{\varepsilon_1 + \varepsilon_2, q}$ as the scalar $q, -q^{-1}$ respectively.

Hence $\mu_{r,q}$ factors through $\nu_r : K\mathcal{B}_r / \langle (R_1 - q)(R_1 + q^{-1}) \rangle \rightarrow H_r(q) \rightarrow \text{End}_{U_q}(V_q^\otimes r)$.

For any $r$, let $a_q(r) = \sum_{w \in \text{Sym}_r} (-q)^{-\ell(w)} T_w \in H_r(q)$ (The $H_r(q)$-alternator). $T_w =$standard basis element of $H_r(q)$

SFT: $\text{Ker}(\nu_r) = \langle a_q(n+1) \rangle$. $\nu_r$ is an isomorphism if $r < n + 1$. 
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Classical theory-orthogonal and symplectic cases

Let $V = \mathbb{C}^n$, $(-, -)$ a non-degenerate symmetric or skew symmetric form on $V$.

$G = O(V)$ or $Sp(V)$, the isometry group of $V$, $(-, -)$.

Let $(b_i), (b'_i)$ be dual bases of $V$, so $(b'_i, b_j) = \delta_{ij}$.

Define $c_0 := \sum_i b_i \otimes b'_i$. Then $c_0$ is independent of basis, and $\in (V \otimes V)^G$.

Define $e \in \text{End}_G(V \otimes^2)$ by $e(v \otimes w) = (v, w)c_0$, and $e_i \in \text{End}_G(V \otimes^r)$ as $e$, acting on factors $(i, i + 1)$ of $V \otimes^r$.

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Brauer showed (1937) that if the diagrams $s_i$ and $f_i$ are as shown:

Figure: $s_i(\mapsto \varepsilon r_i), f_i(\mapsto e_i)$

then the endomorphisms they represent satisfy the same composition laws as the diagrams under concatenation, provided free circles are replaced by $\varepsilon n$.

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![Diagrams](image)

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\begin{array}{ccccccc}
\cdots & \times & \cdots & , & \cdots & \times & \cdots \\
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This shows that there is a homomorphism 
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**FFT (Brauer 1937):** \( \eta_r : B_r(\varepsilon n) \rightarrow \text{End}_G(V^\otimes r) \) is surjective

This may be proved using the FFT for type A, together with a density argument, based on the fact that \( G \) is birationally equivalent to \( \mathbb{A}^{\dim G} \). (cf. Atiyah-Bott-Patodi 1972.)

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This question is complicated by the fact that \( B_r(\varepsilon n) \) is not usually semisimple.

In fact: Rui and Si have shown that: \( B_r(\varepsilon n) \) is semisimple iff \( r \leq d + 1 \) where \( d = n \) if \( G = O(V) \) and \( d = \frac{n}{2} \) is \( G = \text{Sp}(V) \).

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Brauer used this to prove in 1937 the following form of the FFT.

FFT (Brauer 1937): \( \eta_r : B_r(\varepsilon n) \to \text{End}_G(V \otimes r) \) is surjective

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Determine the kernel of $\eta_r : B_r(\varepsilon n) \to \text{End}_G(V^\otimes r)$.

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Theorem (L-Ruibin Zhang)(SFT): There is a quasi-idempotent $\Psi \in B_{d+1}(\varepsilon n)$, explicitly described in terms of diagrams, such that $\text{Ker}(\eta_r)$ is the ideal of $B_r(\varepsilon n)$ generated by $\Psi$. (Recall $d = n$ if $G = O(V)$ and $d = \frac{n}{2}$ is $G = \text{Sp}(V)$.)

To describe $\Psi$ we will need the element $\Sigma_\varepsilon(r) \in B_r(\varepsilon n)$.

\[\Sigma_\varepsilon(r) \in B_r(\varepsilon n) := \sum_{w \in \text{Sym}_r} (-\varepsilon)^{\ell(w)} w.\]

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The symplectic case

The answer is easiest to describe when $G = \text{Sp}_n = \text{Sp}_{2d}$.

Let $\Psi = \sum_{\text{ALL Brauer diagrams } D \in B_{d+1}(-2d)} D$

Then $\Psi$ has the following properties (L-Zhang, arXiv):

- $\Psi^2 = (d + 1)! \Psi$
- $\Psi f_i = f_i \Psi = 0 \ \forall i, 1 \leq i \leq d$
- $\Psi \pi = \pi \Psi \ \forall \pi \in \text{Sym}_{d+1}$ Thus $\frac{1}{(d+1)!} \Psi$ is the central idempotent in $B_{d+1}(-2d)$ corresponding to the trivial representation.
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Specifically, we have

$$\psi = \sum_{k=0}^{\lfloor \frac{d+1}{2} \rfloor} a_k \Xi_k$$

where

$$a_k = \frac{1}{(2^k k!)^2 (d + 1 - 2k)!}.$$

and $\Xi_k$ ($k = 0, 1, \ldots, \lfloor \frac{d+1}{2} \rfloor$) is given by

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The orthogonal case

Let \( G = O(V) = O_n(\mathbb{C}) \).

Let \( \Psi = E_{\frac{n+1}{2}} \), where, for \( p = 0, 1, 2, \ldots, n + 1 \), \( E_p \) is defined diagramatically as

\[
E_{n+1-p} = \Sigma_+(n+1)
\]

Properties of \( \Psi \) (L-Zhang, arXiv, Ann Math 2012):

- \( \Psi^2 = ([n + 1])! (n + 1 - [n + 1])! \Psi \)
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The proof again requires other characterisations of \( \Psi \), and computations in the Brauer algebra.
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The orthogonal case

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Let $\Psi = E_{[\frac{n+1}{2}]}$, where, for $p = 0, 1, 2, \ldots, n+1$, $E_p$ is defined diagramatically as

\[
E_{n+1-p} = \sum_+(n+1)
\]

Properties of $\Psi$ (L-Zhang, arXiv, Ann Math 2012):

- $\Psi^2 = ([\frac{n+1}{2}])!(n+1 - [\frac{n+1}{2}])!\Psi$
- $\Psi f_i = f_i \Psi = 0 \ \forall i, 1 \leq i \leq n$
- $\Psi$ generates $\text{Ker}(\eta_r) \ \forall r$

The proof again requires other characterisations of $\Psi$, and computations in the Brauer algebra.
The quantum case-types $B$, $C$ and $D$.

Let $K = \mathbb{C}(q)$, $U_q = U_q(g)$ where $g$ is the Lie algebra of relevant type; $A$ is the subring of $K$ consisting of functions with no pole at 1.

If $V_q \simeq K^n$ is the ‘natural’ $U_q$-module, then (in the usual notation for weights) $V_q = L_{\varepsilon_1}$ and

$$V_q \otimes V_q = L_{2\varepsilon} \oplus L_{\varepsilon_1+\varepsilon_2} \oplus L_0.$$  

The eigenvalues of $R$ on these 3 summands are respectively $q, -q^{-1}, \varepsilon q^{\varepsilon-n}$.

Hence in the homomorphism $\mu_{r,q} : KB_r \to \text{End}_{U_q}(V_q^\otimes r)$, the images $\sigma$ of the generators of the braid group satisfy the cubic relation $(\sigma - q)(\sigma + q^{-1})(\sigma - \varepsilon q^{\varepsilon-n}) = 0$. 
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One deduces easily that $\mu_{r,q}$ factors through the BMW algebra with appropriate parameters; i.e. we have

$$\nu_{r,q} : \text{BMW}_r(\epsilon q^{\epsilon-n}, q - q^{-1}) \rightarrow \text{End}_{U_q}(V_q^\otimes r).$$

Theorem (FFT): $\nu_{r,q}$ is surjective.

This is proved using an $\mathcal{A}$-form of all structures involved to reduce to the classical case by taking $\lim_{q \rightarrow 1}$.

What of the SFT? The key to using $\lim_{q \rightarrow 1}$ is the cellular structure of both $\text{BMW}_r(\epsilon q^{\epsilon-n}, q - q^{-1})$ and $B_r(\epsilon n)$.

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What of the SFT? The key to using $\lim_{q \to 1}$ is the cellular structure of both $BMW_r(\varepsilon q^{\varepsilon - n}, q - q^{-1})$ and $B_r(\varepsilon n)$.

Let $BMW_r^\varepsilon(q)$ be the BMW algebra over $A$ with parameters $\varepsilon q^{\varepsilon - n}, q - q^{-1}$. Then:
\( BMW_r^\varepsilon(q) \) and \( B_r(\varepsilon n) \) have cellular structures with the same cell datum, and writing \( \lim_{q \to 1} (-) = - \otimes_A \mathbb{C} \), we have \( \lim_{q \to 1} BMW_r^\varepsilon(q) = B_r(\varepsilon n) \).

All cell modules \( W(\lambda) \) of \( B_r(\varepsilon n) \) are of the form \( W(\lambda) = \lim_{q \to 1} W_q(\lambda) \), where \( W_q(\lambda) \) is the corresponding cell module for \( BMW_r^\varepsilon(q) \).

This leads to the following situation.

\[
\begin{array}{cccc}
0 & \rightarrow & \text{Ker}(\nu_{q,r}) & \rightarrow & BMW_r^\varepsilon(K) & \nu_{q,r} & \rightarrow & \text{End}_{U_q}(V_q^\otimes r) & \rightarrow & 0 \\
& & \uparrow{-}\otimes_A K & & \uparrow{-}\otimes_A K & & & \uparrow{-}\otimes_A K & & \\
0 & \rightarrow & \text{Ker}(\nu_{A,r}) & \rightarrow & BMW_r^\varepsilon(q) & \nu_{A,r} & \rightarrow & \text{End}_{U_A}(V_A^\otimes r) & \rightarrow & 0 \\
& & \downarrow{\lim_{q \to 1}} & & \downarrow{\lim_{q \to 1}} & & & \downarrow{\lim_{q \to 1}} & & \\
0 & \rightarrow & \text{Ker}(\nu_r) & \rightarrow & B_r(\varepsilon n) & \nu_r & \rightarrow & \text{End}_G(V^\otimes r) & \rightarrow & 0 \\
\end{array}
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\( \text{BMW}_r^\varepsilon(q) \) and \( B_r(\varepsilon n) \) have cellular structures with the same cell datum, and writing \( \lim_{q \to 1}(-) = - \otimes_A \mathbb{C} \), we have \( \lim_{q \to 1} \text{BMW}_r^\varepsilon(q) = B_r(\varepsilon n) \).

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\[
\begin{align*}
0 & \to \text{Ker}(\nu_{q,r}) & \to & \text{BMW}_r^\varepsilon(K) & \nu_{q,r} & \to & \text{End}_{U_q}(V_q^\otimes r) & \to & 0 \\
& & \uparrow_{- \otimes_A K} & & & & \uparrow_{- \otimes_A K} & & \\
0 & \to \text{Ker}(\nu_{A,r}) & \to & \text{BMW}_r^\varepsilon(q) & \nu_{A,r} & \to & \text{End}_{U_A}(V_A^\otimes r) & \to & 0 \\
& & \downarrow_{\lim_{q \to 1}} & & & & \downarrow_{\lim_{q \to 1}} & & \\
0 & \to \text{Ker}(\nu_r) & \to & B_r(\varepsilon n) & \nu_r & \to & \text{End}_G(V^\otimes r) & \to & 0
\end{align*}
\]
\( \textit{BMW}_r(\varepsilon) \) and \( B_r(\varepsilon\eta) \) have cellular structures with the same cell datum, and writing \( \lim_{q \to 1}(-) = - \otimes A \mathbb{C} \), we have \( \lim_{q \to 1} \textit{BMW}_r(\varepsilon) = B_r(\varepsilon\eta) \).

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This leads to the following situation.

\[
\begin{array}{cccccc}
0 & \to & \text{Ker}(\nu_{q,r}) & \to & \text{BMW}_r(\varepsilon)(K) & \xrightarrow{\nu_{q,r}} & \text{End}_{U_q}(V_q \otimes r) & \to & 0 \\
& & \uparrow{-}\otimes A K & & \uparrow{-}\otimes A K & & \uparrow{-}\otimes A K \\
0 & \to & \text{Ker}(\nu_{A,r}) & \to & \text{BMW}_r(\varepsilon)(q) & \xrightarrow{\nu_{A,r}} & \text{End}_{U_A}(V_A \otimes r) & \to & 0 \\
& & \downarrow{\lim_{q \to 1}} & & \downarrow{\lim_{q \to 1}} & & \downarrow{\lim_{q \to 1}} \\
0 & \to & \text{Ker}(\nu_r) & \to & B_r(\varepsilon\eta) & \xrightarrow{\nu_r} & \text{End}_G(V \otimes r) & \to & 0
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\( \text{BMW}_r^\varepsilon(q) \) and \( B_r(\varepsilon n) \) have cellular structures with the same cell datum, and writing \( \lim_{q \to 1} (-) = - \otimes_A \mathbb{C} \), we have \( \lim_{q \to 1} \text{BMW}_r^\varepsilon(q) = B_r(\varepsilon n) \).

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This leads to the following situation.

\[
\begin{array}{ccccccc}
0 & \rightarrow & \text{Ker}(\nu_{q,r}) & \rightarrow & \text{BMW}_r^\varepsilon(K) & \rightarrow & \text{End}_U q(V_q \otimes r) & \rightarrow & 0 \\
& & \uparrow - \otimes_A K & & \uparrow - \otimes_A K & & \uparrow - \otimes_A K & & \\
0 & \rightarrow & \text{Ker}(\nu_{A,r}) & \rightarrow & \text{BMW}_r^\varepsilon(q) & \rightarrow & \text{End}_U A(V_A \otimes r) & \rightarrow & 0 \\
& & \downarrow \lim_{q \to 1} & & \downarrow \lim_{q \to 1} & & \downarrow \lim_{q \to 1} & & \\
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\[
\begin{array}{cccccc}
0 & \to & \ker(\nu_{q,r}) & \to & \text{BMW}_r (K) & \nu_{q,r} \to & \text{End}_{U_q} (V^\otimes r) & \to & 0 \\
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From this one deduces:

**Theorem (LZ):** Let \( \psi \in B_r(\varepsilon n) \) be such that \( \text{Ker}\, \nu_r = \langle \psi \rangle \).

Assume that \( \psi_q \in BMW_\varepsilon(q) \) is such that

- \( \psi_q^2 = f(q)\psi_q \) where \( f(1) \neq 0 \)
- \( \lim_{q \to 1} \psi_q = c\psi \), with \( c \neq 0 \).

Then \( \text{Ker}(\nu_{r,q}) = \langle \psi_q \rangle \).

**Cor:** \( \text{Ker}(\nu_{r,q}) = \langle \psi_q \rangle \) for some quasi-idempotent \( \psi_q \in BMW_\varepsilon_{d+1}(q) \) in both the symplectic and orthogonal cases.

In the symplectic case, we may take \( \psi_q \) to be the idempotent corresponding to the ‘trivial representation’ of \( BMW_\varepsilon_{d+1}(q) \).

In the orthogonal case we use an argument about lifting idempotents.

**Remark:** Jun Hu and Xsiao have proved that the above statements may be generalised to all characteristics, and have proved a linear version of the quantum statement.
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**Theorem (LZ):** Let $\psi \in B_r(\varepsilon_n)$ be such that $\text{Ker}\,\nu_r = \langle \psi \rangle$. Assume that $\psi_q \in BMW_\varepsilon^r(q)$ is such that

- $\psi_q^2 = f(q)\psi_q$ where $f(1) \neq 0$
- $\lim_{q \to 1} \psi_q = c\psi$, with $c \neq 0$.

Then $\text{Ker}(\nu_r,q) = \langle \psi_q \rangle$.

**Cor:** $\text{Ker}(\nu_r,q) = \langle \psi_q \rangle$ for some quasi-idempotent $\psi_q \in BMW_{d+1}^\varepsilon(q)$ in both the symplectic and orthogonal cases.

In the symplectic case, we may take $\psi_q$ to be the idempotent corresponding to the ‘trivial representation’ of $BMW_{d+1}^\varepsilon(q)$.

In the orthogonal case we use an argument about lifting idempotents.

**Remark:** Jun Hu and Xsiao have proved that the above statements may be generalised to all characteristics, and have proved a linear version of the quantum statement.
Superalgebras

Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space.

If $\dim V_0 = m$, $\dim V_1 = n$, say that $\text{sdim } V = (m|n)$.

Suppose $V$ has an even non-degenerate bilinear form $(-, -)$ which is symmetric on $V_0$, skew symmetric on $V_1$, and satisfies $(V_0, V_1) = (V_1, V_0) = 0$. So $\text{sdim } V = (m|2n)$. This is an orthosymplectic superspace.

If $V, W$ are $\mathbb{Z}_2$-graded, so are $V^*$ and $\text{Hom}_\mathbb{C}(V, W) \cong W \otimes_\mathbb{C} V^*$. In particular, so is $\text{End}_\mathbb{C}(V)$.

- If $\text{sdim } (V) = (m|n)$ the general linear supergroup $\mathfrak{gl}(V) = \mathfrak{gl}(m|n)$ is the $\mathbb{Z}_2$-graded Lie algebra $\text{End}_\mathbb{C}(V)$, with Lie product $[X, Y] = XY - (-1)^{[X][Y]} YX$.

- The orthosymplectic Lie algebra $\mathfrak{osp}(m|2n)$ is the $\mathbb{Z}_2$-graded subalgebra of $\mathfrak{gl}(m|2n)$ defined by
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  \{ X \in \mathfrak{gl}(m|2n) \mid (Xv, w) + (-1)^{[X][v]}(v; Xw) = 0 \}.
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The Lie superalgebra $\mathfrak{gl}(V)$ acts on $V^\otimes r$ via

$$X.v_1 \otimes \ldots \otimes v_r = \sum_{i=1}^{r} (-1)^{[X][v_1]+\cdots+[v_{i-1}]} v_1 \otimes \ldots \otimes Xv_i \otimes \ldots \otimes v_r.$$ 

The subalgebra $\mathfrak{osp}(m|2n)$ acts correspondingly on $V^\otimes r$. Further the group $G := O(V_0) \times Sp(V_1)$ also acts on $V^\otimes r$, compatibly with $\mathfrak{osp}$.

We have the endomorphisms $\tau, e \in \text{End}_{\mathfrak{osp},G}(V \otimes V)$:

$$\tau(v \otimes w) = (-1)^{[v][w]} w \otimes v$$

$e$ is defined in a similar way to the classical orthogonal and symplectic cases, using dual homogeneous bases of $V$.

This shows that: the Brauer algebra $B_r(m - 2n)$ acts on $V^\otimes r$. This action commutes with that of $\mathfrak{osp}$ and of $G$. 
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This leads to:

**Theorem (LZ 2013, see also Serge’ev)** The map \( B_r(m - 2n) \to \text{End}_{osp,G}(V^\otimes r) \) is surjective.

The proof is by converting to an equivalent statement for the orthosymplectic supergroup over an infinite dimensional Grassmann algebra, and using the geometric method of Atiyah et al.

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Further questions

Integral versions of all cases; analysis at roots of unity; tilting modules

For which pairs $g, V$ do we have $\mathcal{A}B_r \to \text{End}_{U^A(g)}(V \otimes r)$ surjective? And for which subrings $A$ of $K$?

When does the above map factor through a cellular algebra? (cf. ALZ)
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Richard Brauer, “On algebras which are connected with the semisimple continuous groups”, Ann. of Math. (2) 38 (1937), 857–872.


